

## STATIC AND DYNAMIC ANALYSES OF COMPOSITE BEAMS WITH INTERLAYER SLIP

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**Abstract.** The present paper gives analytical solutions for shear deformable two-layer beams with weak shear connections. Timoshenko's kinematic assumptions are applied to both layers with different cross-sectional rotations. A linear constitutive equation is used between the horizontal shear force and the interlayer slip. The applied loads act in the plane of symmetry of a two-layered beam and the material-geometrical properties do not depend on the axial coordinate. For simply supported beam closed form solutions are derived for the deflection, slip and cross-sectional rotations. The eigenfrequencies of a simply supported beam are determined and compared with the solutions obtained by the applications of Euler-Bernoulli and Euler-Bernoulli-Rayleigh beam models.

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*Keywords:* interlayer slip, shear connection, composite beam, free vibration

### 1. INTRODUCTION

Layered beams made of different elastic materials are frequently used in construction and they have created a growing interest in the different engineering sectors where both high strength-to-weight and stiffness-to-weight ratio are desired. There are many ways to form a connection between layers made of different materials. In some cases it occurs that the connection is weak in shear permitting only a relative slip, but preserving the contact in normal direction. The problem of layered beams with deformable shear connections has been studied for a long time. The first theories for these composite beams were developed by Granholm [1], Pleskov [2], Stüssi [3] and Newmark et al. [4]. The static analysis done by Newmark et al. [4] is based on the Euler-Bernoulli beam theory and has become a basis for investigating layered beam systems with interlayer slip [5–22]. Today the analytical and numerical FEM solution are refined [5–26]. Several studies [5–8, 13] present FEM solutions for multilayer beams with weak shear connections using the Euler-Bernoulli beam theory. Exact first and second order static analyses for composite beam-columns with partial shear interaction subjected to transverse and axial loads are given by Girhammar and Gopu [9]. By using variational methods Girhammar and Pan [10] derive ordinary differential

equations for the deflections and set up the corresponding boundary conditions for partially composite Euler-Bernoulli beams and beam-columns. A simplified analysis and design method for composite members with partial shear interaction that predicts the deflections and stresses has been proposed by Girhammar [11]. In [16,17,20] researchers developed FEM formulations for composite beams with deformable shear connection. The derived stiffness matrix takes the effects of interface slip and shear deformations into account. In [20] it is assumed that the cross-sectional rotations are not the same for the different beam components and the effect of shear connectors on a composite beam element is described by two springs which are separately placed at the two ends of the considered element.

Dynamic analysis of a composite beam with deformable shear connections based on the Euler-Bernoulli beam theory is presented by Girhammar and Gopu [23]. They consider free and forced vibrations. The governing differential equations and corresponding boundary conditions are derived for partial-interaction composite members and exact analytical solution for simply supported boundary conditions is presented in [25]. In paper [24] an analytical solution for free vibrations of shear-deformable two-layer beams with interlayer slip and axial load is developed. The effect of transverse shear flexibility of two layers is taken into account in a general way. Each layer behaves as a Timoshenko beam.

The present paper deals with two-layer beams with interlayer slip and gives analytical solutions for deflection, slip and cross-sectional rotations in the case of a static equilibrium problem. By introducing the inertia forces into the equilibrium equations we derive the equations of free vibrations as well. Applications of the equations we have established are illustrated via numerical examples.

## 2. GOVERNING EQUATIONS

In the reference configuration a composite beam with two components occupies the cylindrical region  $B = A \times [0, L]$  generated by translating its cross-section  $A$  with a regular boundary  $\partial A$  along a rectilinear axis, normal to the cross-section. The cross-section  $A$  is divided into two parts  $A_1$  and  $A_2$  by the curve  $\partial A_{12}$  describing the positions of continuous connection such that (see Figure 1)

$$B_i = A_i \times (0, L), \quad (i = 1, 2), \quad A = A_1 \cup A_2, \quad B = B_1 \cup B_2, \quad (2.1)$$

$$\partial A_i = \partial A_{i0} \cup \partial A_{12}, \quad (i = 1, 2), \quad \partial A = \partial A_{10} \cup \partial A_{12}. \quad (2.2)$$

Here  $L$  is the length of the beam. A point  $P$  in  $\bar{B} = B \cup \partial B$  ( $\partial B$  is the boundary surface of  $B$ ) is indicated by the position vector  $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ , where  $x, y, z$  and  $\mathbf{e}_m$  ( $|\mathbf{e}_m| = 1, m = x, y, z$ ) refer to a rectangular coordinate system  $Oxyz$ . The equation of the common boundary surface of the beam components  $B_1$  and  $B_2$  is  $y = 0, 0 \leq z \leq L$  (see Figure 1). The center of  $A_i$  is denoted by  $C_i$  ( $i = 1, 2$ ). The plane  $yz$  is the plane of symmetry for the geometrical and material properties and loading conditions. According to Timoshenko's beam theory, which is valid for each homogeneous layer, the deformation of the beam is described by the following displacement field (see Figure 2)

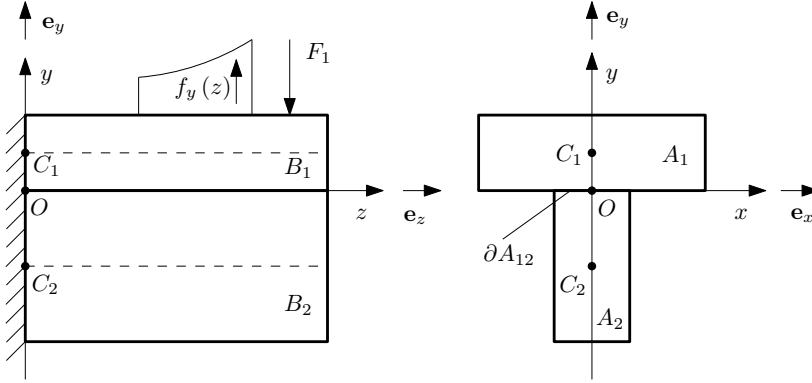


Figure 1. A composite beam with weak shear connection

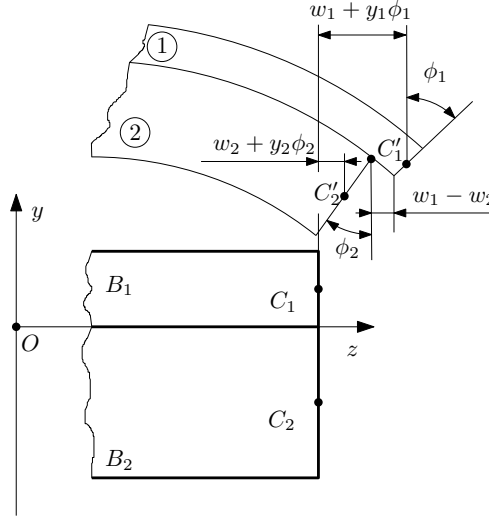


Figure 2. Kinematical model for a composite beam with interlayer slip

$$\mathbf{u} = u(x, y, z) \mathbf{e}_x + v(x, y, z) \mathbf{e}_y + w(x, y, z) \mathbf{e}_z, \quad (2.3)$$

where

$$u = 0, \quad v = v(z), \quad w(y, z) = w_i(z) + y\phi_i(z), \quad (x, y, z) \in B_i, \quad (i = 1, 2). \quad (2.4)$$

In Figure 2  $C'_i$  ( $i = 1, 2$ ) denotes the center of  $A_i$  in the deformed configuration of the considered cross-section. On the common boundary of the beam component the axial displacement  $w$  has a jump which is called an interlayer slip. From equation (2.4) and the definition of interlayer slip  $s$  it follows that

$$s(x, y, z) = w_1(z) - w_2(z), \quad y = 0, \quad 0 \leq z \leq L. \quad (2.5)$$

Application of the strain displacement relationships of the linearized theory of elasticity yields [27, 28]

$$\varepsilon_x = \varepsilon_y = \gamma_{xy} = \gamma_{xz} = 0, \quad (2.6)$$

$$\varepsilon_z = \frac{dw_i}{dz} + y \frac{d\phi_i}{dz}, \quad (x, y, z) \in B_i, \quad (i = 1, 2), \quad (2.7)$$

$$\gamma_{yz} = \frac{dv}{dz} + \phi_i, \quad (x, y, z) \in B_i, \quad (i = 1, 2), \quad (2.8)$$

where  $\varepsilon_x, \varepsilon_y, \varepsilon_z$  and  $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$  are the axial and shearing strains. From Hooke's law we get the normal stress  $\sigma_z$  and the shearing stress  $\tau_{yz}$  in terms of strains as

$$\sigma_z = E_i \left( \frac{dw_i}{dz} + y \frac{d\phi_i}{dz} \right), \quad (x, y, z) \in B_i, \quad (i = 1, 2), \quad (2.9)$$

$$\tau_{yz} = G_i \left( \frac{dv}{dz} + \phi_i \right), \quad (x, y, z) \in B_i, \quad (i = 1, 2). \quad (2.10)$$

In equations (2.9), (2.10)  $E_i$  is the Young modulus and  $G_i$  is the shear modulus. We introduce the following section forces and section moment

$$N_i = \int_{A_i} \sigma_z dA = A_i E_i \left( \frac{dw_i}{dz} + y_i \frac{d\phi_i}{dz} \right), \quad y_i = \frac{1}{A_i} \int_{A_i} y dA, \quad (i = 1, 2), \quad (2.11)$$

$$M_i = \int_{A_i} y \sigma_z dA = A_i E_i \left( y_i \frac{dw_i}{dz} + g_i^2 \frac{d\phi_i}{dz} \right), \quad g_i^2 = \frac{1}{A_i} \int_{A_i} y^2 dA, \quad (i = 1, 2), \quad (2.12)$$

$$V_i = \kappa_i A_i G_i \left( \frac{dv}{dz} + \phi_i \right), \quad (i = 1, 2), \quad (2.13)$$

where  $\kappa_i$  is the shear factor of the cross-section  $A_i$  ( $i = 1, 2$ ) [29]. The analysis of the composite beam with interlayer slip is restricted to the case of absent axial force  $N$ , that is, we have

$$N = N_1 + N_2 = A_1 E_1 \left( \frac{dw_1}{dz} + y_1 \frac{d\phi_1}{dz} \right) + A_2 E_2 \left( \frac{dw_2}{dz} + y_2 \frac{d\phi_2}{dz} \right) = 0. \quad (2.14)$$

From equations (2.5), (2.14) it follows that

$$\frac{dw_1}{dz} = \frac{A_2 E_2}{\langle AE \rangle} \frac{ds}{dz} - y_1 \frac{A_1 E_1}{\langle AE \rangle} \frac{d\phi_1}{dz} - y_2 \frac{A_2 E_2}{\langle AE \rangle} \frac{d\phi_2}{dz}, \quad (2.15)$$

$$\frac{dw_2}{dz} = -\frac{A_1 E_1}{\langle AE \rangle} \frac{ds}{dz} - y_1 \frac{A_1 E_1}{\langle AE \rangle} \frac{d\phi_1}{dz} - y_2 \frac{A_2 E_2}{\langle AE \rangle} \frac{d\phi_2}{dz}. \quad (2.16)$$

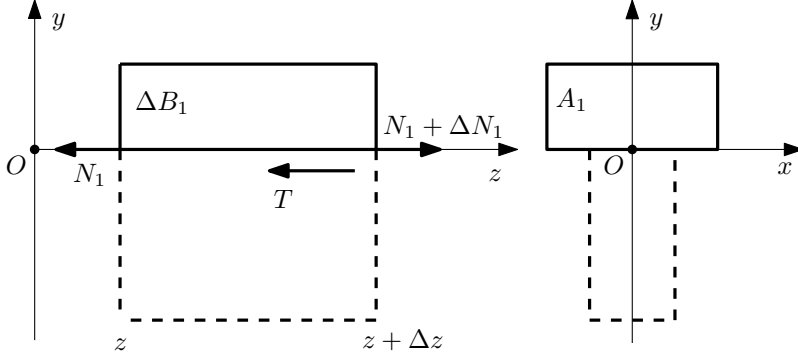


Figure 3. Equilibrium condition in the direction  $z$  for a small beam element

Here

$$\langle AE \rangle = A_1 E_1 + A_2 E_2. \quad (2.17)$$

It is obvious that

$$N_1 = \langle AE \rangle_{-1} \left[ \frac{ds}{dz} + y_1 \frac{d\phi_1}{dz} - y_2 \frac{d\phi_2}{dz} \right], \quad (2.18)$$

$$N_2 = \langle AE \rangle_{-1} \left[ -\frac{ds}{dz} - y_1 \frac{d\phi_1}{dz} + y_2 \frac{d\phi_2}{dz} \right], \quad (2.19)$$

$$M_1 = \langle AE \rangle_{-1} \left[ y_1 \frac{ds}{dz} + c_1 \frac{d\phi_1}{dz} - y_1 y_2 \frac{d\phi_2}{dz} \right], \quad (2.20)$$

$$M_2 = \langle AE \rangle_{-1} \left[ -y_2 \frac{ds}{dz} - y_1 y_2 \frac{d\phi_1}{dz} + c_2 \frac{d\phi_2}{dz} \right], \quad (2.21)$$

where

$$\frac{1}{\langle AE \rangle_{-1}} = \frac{1}{A_1 E_1} + \frac{1}{A_2 E_2}, \quad (2.22)$$

$$c_1 = \frac{A_1 E_1}{\langle AE \rangle_{-1}} g_1^2 - y_1^2 \frac{A_1 E_1}{A_2 E_2}, \quad (2.23)$$

$$c_2 = \frac{A_2 E_2}{\langle AE \rangle_{-1}} g_2^2 - y_2^2 \frac{A_2 E_2}{A_1 E_1}. \quad (2.24)$$

Application of the equilibrium condition for the axial forces in the beam component  $B_1$  yields (see Figure 3)

$$\frac{dN_1}{dz} - T = 0. \quad (2.25)$$

In equation (2.25)  $T$  is the interlayer shear force. It is assumed that  $T$  is a linear function of the interlayer slip, that is

$$T = ks, \quad (2.26)$$

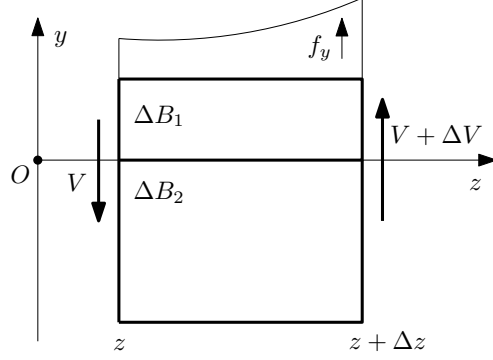


Figure 4. Shear force and vertical load on a small beam element  $\Delta B = \Delta B_1 \cup \Delta B_2$

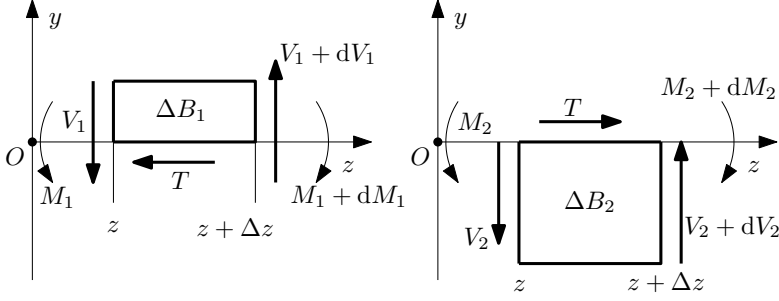


Figure 5. Shear forces and bending moments on beam elements  $\Delta B_1$  and  $\Delta B_2$

where  $k$  is the slip modulus. From Figures 4 and Figure 5 we have the following equilibrium equations:

$$\frac{dV}{dz} + f_y = 0, \quad V = V_1 + V_2, \quad (2.27)$$

$$\frac{dM_1}{dz} - V_1 = 0, \quad (2.28)$$

$$\frac{dM_2}{dz} - V_2 = 0. \quad (2.29)$$

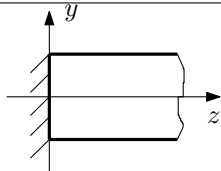
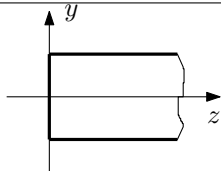
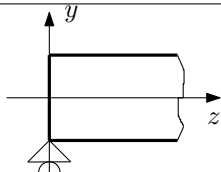
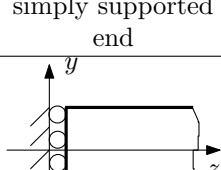
In order to formulate the possible boundary conditions we shall consider the virtual work of the section forces and section moments on a kinematically admissible displacement field

$$\tilde{\mathbf{u}}_i = \tilde{v}(z) \mathbf{e}_y + \left( \tilde{w}_i(z) + y \tilde{\phi}_i(z) \right) \mathbf{e}_z, \quad (x, y, z) \in B_i, \quad (i = 1, 2). \quad (2.30)$$

A detailed computation results in

$$W = \int_{A_1} \sigma_z \tilde{w}_1(z) dA + \int_{A_2} \sigma_z \tilde{w}_2(z) dA + \int_{A_1} y \sigma_z \tilde{\phi}_1(z) dA + \int_{A_2} y \sigma_z \tilde{\phi}_2(z) dA +$$

Table 1. Classical boundary conditions

Type	Boundary condition
 <p>fixed end</p>	$v = 0, s = 0, \phi_1 = 0, \phi_2 = 0$ (kinematical boundary conditions)
 <p>free end</p>	$N_1 = 0, V = 0, M_1 = 0, M_2 = 0$ (forced boundary conditions)
 <p>simply supported end</p>	$v = 0, N_1 = 0, M_1 = 0, M_2 = 0$ (mixed boundary conditions)
 <p>guided end</p>	$s = 0, \phi_1 = 0, \phi_2 = 0, V = 0$ (mixed boundary conditions)

$$+ \int_A \tau_{yz} \tilde{v}(z) dA = N_1 \tilde{w}_1 + N_2 \tilde{w}_2 + M_1 \tilde{\phi}_1 + M_2 \tilde{\phi}_2 + V \tilde{v} = N_1 \tilde{s} + M_1 \tilde{\phi}_1 + M_2 \tilde{\phi}_2 + V \tilde{v} \quad (2.31)$$

where

$$\tilde{s} = \tilde{w}_1(z) - \tilde{w}_2(z). \quad (2.32)$$

From equation (2.31) we obtain the possible combinations of the boundary conditions at the end cross-section

$$V \text{ or } v \text{ may be prescribed,} \quad (2.33)$$

$$N_1 \text{ or } s \text{ may be prescribed,} \quad (2.34)$$

$$M_1 \text{ or } \phi_1 \text{ may be prescribed,} \quad (2.35)$$

$$M_2 \text{ or } \phi_2 \text{ may be prescribed.} \quad (2.36)$$

We remark that some classical boundary conditions are listed on the basis of equations (2.33–2.36) in Table 1. It can be checked by utilizing equations (2.18) and (2.20) that the boundary conditions for a simply supported end and a free end are

$$\frac{ds}{dz} = 0, \quad \frac{d\phi_1}{dz} = 0, \quad \frac{d\phi_2}{dz} = 0. \quad (2.37)$$

The combination of equations (2.13), (2.18), (2.20), (2.21), (2.26) with equations (2.25), (2.27), (2.28), (2.29) yields a system of linear differential equations for the functions  $v = v(z)$ ,  $s = s(z)$ ,  $\phi_1 = \phi_1(z)$  and  $\phi_2 = \phi_2(z)$

$$\frac{d^2s}{dz^2} + y_1 \frac{d^2\phi_1}{dz^2} - y_2 \frac{d^2\phi_2}{dz^2} - \frac{k}{\langle AE \rangle_{-1}} s = 0, \quad (2.38)$$

$$y_1 \frac{d^2s}{dz^2} + c_1 \frac{d^2\phi_1}{dz^2} - y_1 y_2 \frac{d^2\phi_2}{dz^2} - \kappa_1 \frac{G_1 A_1}{\langle AE \rangle_{-1}} \left( \frac{dv}{dz} + \phi_1 \right) = 0, \quad (2.39)$$

$$-y_2 \frac{d^2s}{dz^2} - y_1 y_2 \frac{d^2\phi_1}{dz^2} + c_2 \frac{d^2\phi_2}{dz^2} - \kappa_2 \frac{G_2 A_2}{\langle AE \rangle_{-1}} \left( \frac{dv}{dz} + \phi_2 \right) = 0, \quad (2.40)$$

$$\kappa_1 G_1 A_1 \left( \frac{d^2v}{dz^2} + \frac{d\phi_1}{dz} \right) + \kappa_2 G_2 A_2 \left( \frac{d^2v}{dz^2} + \frac{d\phi_2}{dz} \right) = -f_y(z). \quad (2.41)$$

### 3. SOLUTION FOR A SIMPLY SUPPORTED BEAM

For a simply supported beam we look for the solution of the ordinary differential equation system (2.38–2.41) in the following form:

$$v(z) = \sum_{j=1}^{\infty} v_j \sin j \frac{\pi}{L} z, \quad s(z) = \sum_{j=1}^{\infty} s_j \cos j \frac{\pi}{L} z, \quad (3.1a)$$

$$\phi_1(z) = \sum_{j=1}^{\infty} \phi_{1j} \cos j \frac{\pi}{L} z, \quad \phi_2(z) = \sum_{j=1}^{\infty} \phi_{2j} \cos j \frac{\pi}{L} z. \quad (3.1b)$$

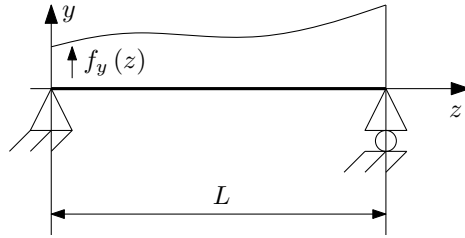


Figure 6. Simply supported beam

These functions satisfy the boundary conditions

$$v = 0, \quad N_1 = 0, \quad M_1 = 0, \quad M_2 = 0 \quad (3.2)$$



for arbitrary values of  $v_j$ ,  $s_j$ ,  $\phi_{1j}$ ,  $\phi_{2j}$ , ( $j = 1, 2, \dots$ ) at both ends of the beam. Substitution of  $v = v(z)$ ,  $s = s(z)$ ,  $\phi_1 = \phi_1(z)$  and  $\phi_2 = \phi_2(z)$  into the ODE system (2.38–2.41) results in a system of linear equations for  $v_j$ ,  $s_j$ ,  $\phi_{1j}$ ,  $\phi_{2j}$ , ( $j = 1, 2, \dots$ ). This can be written as

$$\mathbf{C}_j \mathbf{x}_j = \mathbf{f}_j, \quad (j = 1, 2, \dots) \quad (3.3)$$

where

$$\mathbf{C}_j = [c_{jpq}], \quad (p, q = 1, 2, 3, 4), \quad (3.4)$$

$$c_{j11} = -\left(j\frac{\pi}{L}\right)^2 - \frac{k}{\langle AE \rangle_{-1}}, \quad c_{j12} = -y_1 \left(j\frac{\pi}{L}\right)^2, \quad c_{j13} = y_2 \left(j\frac{\pi}{L}\right)^2, \quad c_{j14} = 0, \quad (3.5)$$

$$c_{j21} = -y_1 \left(j\frac{\pi}{L}\right)^2, \quad c_{j22} = -c_1 \left(j\frac{\pi}{L}\right)^2 + \frac{\kappa_1 G_1 A_1}{\langle AE \rangle_{-1}},$$

$$c_{j23} = y_1 y_2 \left(j\frac{\pi}{L}\right)^2, \quad c_{j24} = -\frac{\kappa_1 G_1 A_1}{\langle AE \rangle_{-1}} j \frac{\pi}{L}, \quad (3.6)$$

$$c_{j31} = y_2 \left(j\frac{\pi}{L}\right)^2, \quad c_{j32} = y_1 y_2 \left(j\frac{\pi}{L}\right)^2,$$

$$c_{j33} = -c_2 \left(j\frac{\pi}{L}\right)^2 + \frac{\kappa_2 G_2 A_2}{\langle AE \rangle_{-1}}, \quad c_{j34} = -\frac{\kappa_2 G_2 A_2}{\langle AE \rangle_{-1}} j \frac{\pi}{L}, \quad (3.7)$$

$$c_{j41} = 0, \quad c_{j42} = -\kappa_1 G_1 A_1 j \frac{\pi}{L}, \quad c_{j43} = -\kappa_2 G_2 A_2 j \frac{\pi}{L},$$

$$c_{j44} = -(\kappa_1 G_1 A_1 + \kappa_2 G_2 A_2) \left(j\frac{\pi}{L}\right)^2, \quad (3.8)$$

$$\mathbf{x}_j^T = [s_j, \phi_{1j}, \phi_{2j}, v_j], \quad (3.9)$$

$$\mathbf{f}_j^T = [0, 0, 0, f_j]. \quad (3.10)$$

In formula (3.10)  $f_j$  is defined as

$$f_j = \frac{2}{L} \int_0^L f_y(z) \sin j \frac{\pi}{L} z dz, \quad (j = 1, 2, \dots). \quad (3.11)$$

Assume that the beam is subjected to the load

$$f_y(z) = -f \left[ H\left(z - \frac{L}{2} + \frac{l}{2}\right) - H\left(z - \frac{L}{2} - \frac{l}{2}\right) \right] \quad (3.12)$$

where  $H = H(z)$  is the Heaviside function. The type of applied load given by equation (3.12) and the meanings of  $f$  and  $l$  ( $0 \leq l \leq L$ ) are shown in Figure 7. For this load the Fourier coefficients  $f_j$  are given by the following equation:

$$f_j = -\frac{4f}{j\pi} \sin\left(\frac{j\pi}{2}\right) \sin\left(\frac{j\pi}{2} \frac{l}{L}\right), \quad (j = 1, 2, \dots). \quad (3.13)$$

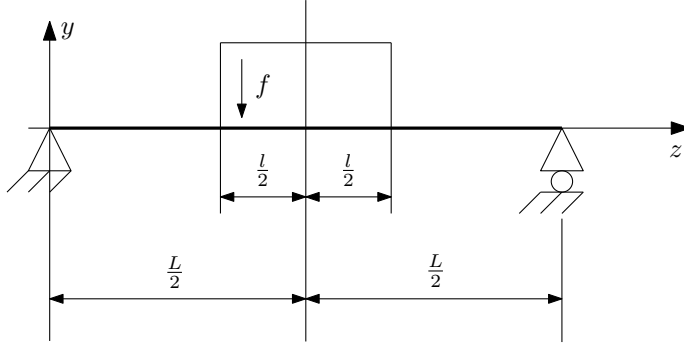


Figure 7. Illustration of applied load

## 4. FREE VIBRATIONS OF A SIMPLY SUPPORTED BEAM

To formulate the governing equations of the free vibrations we introduce the inertia forces into equations (2.27), (2.28) and (2.29). It is obvious that all physical quantities depend on time  $t$  too, that is  $v = v(z, t)$ ,  $s = s(z, t)$ ,  $\phi_1 = \phi_1(z, t)$ ,  $\phi_2 = \phi_2(z, t)$ , etc. Assuming free vibrations we obtain

$$v = \bar{v}(z) \sin \omega t, \quad s = \bar{s}(z) \sin \omega t, \quad \phi_1 = \bar{\phi}_1(z) \sin \omega t, \quad \phi_2 = \bar{\phi}_2(z) \sin \omega t, \quad (4.1)$$

where  $\omega$  is a natural frequency and  $\bar{v}(z), \dots, \bar{\phi}_2(z)$  are the unknown amplitude functions. The inertia force from the vertical motion is of the form

$$f_y(z, t) = -(\rho_1 A_1 + \rho_2 A_2) \frac{\partial^2 v}{\partial t^2} = \omega^2 (\rho_1 A_1 + \rho_2 A_2) \bar{v}(z) \sin \omega t. \quad (4.2)$$

The inertia couples from the cross-sectional rotations are

$$m_1(z, t) = -\rho_1 I_1 \frac{\partial^2 \phi_1}{\partial t^2} = \omega^2 \rho_1 I_1 \bar{\phi}_1(z) \sin \omega t, \quad (4.3)$$

$$m_2(z, t) = -\rho_2 I_2 \frac{\partial^2 \phi_2}{\partial t^2} = \omega^2 \rho_2 I_2 \bar{\phi}_2(z) \sin \omega t. \quad (4.4)$$

In equations (4.2–4.4)  $\rho_i$  ( $i = 1, 2$ ) is the mass density and

$$I_i = \int_{A_i} y^2 dA, \quad (i = 1, 2). \quad (4.5)$$

Upon substitution of equation (4.2) into equation (2.27) we obtain

$$\kappa_1 G_1 A_1 \left( \frac{d\bar{v}}{dz} + \bar{\phi}_1 \right) + \kappa_2 G_2 A_2 \left( \frac{d\bar{v}}{dz} + \bar{\phi}_2 \right) + \omega^2 (\rho_1 A_1 + \rho_2 A_2) \bar{v} = 0. \quad (4.6)$$

Combination of equations (2.28) and (2.29) with equations (4.3) and (4.4) yields the following two equations:

$$y_1 \frac{d^2 \bar{s}}{dz^2} + c_1 \frac{d^2 \bar{\phi}_1}{dz^2} - y_1 y_2 \frac{d^2 \bar{\phi}_2}{dz^2} - \frac{\kappa_1 G_1 A_1}{\langle AE \rangle_{-1}} \left( \frac{d\bar{v}}{dz} + \bar{\phi}_1 \right) + \omega^2 \frac{\rho_1 I_1}{\langle AE \rangle_{-1}} \bar{\phi}_1(z) = 0, \quad (4.7)$$

$$-y_2 \frac{d^2 \bar{s}}{dz^2} - y_1 y_2 \frac{d^2 \bar{\phi}_1}{dz^2} + c_2 \frac{d^2 \bar{\phi}_2}{dz^2} - \frac{\kappa_2 G_2 A_2}{\langle AE \rangle_{-1}} \left( \frac{d\bar{v}}{dz} + \bar{\phi}_2 \right) + \omega^2 \frac{\rho_2 I_2}{\langle AE \rangle_{-1}} \bar{\phi}_2(z) = 0. \quad (4.8)$$

We introduce the mass matrix  $\mathbf{M}$  by the following definition

$$\mathbf{M}_j = [m_{jpp}], \quad (p, q = 1, 2, 3, 4), \quad i = 1, 2, \dots, \quad (4.9)$$

$$m_{j11} = m_{j12} = m_{j13} = m_{j14} = 0, \quad (4.10)$$

$$m_{j21} = m_{j23} = m_{j24} = 0, \quad m_{j22} = -\frac{\rho_1 I_1}{\langle AE \rangle_{-1}}, \quad (4.11)$$

$$m_{j31} = m_{j32} = m_{j34} = 0, \quad m_{j33} = -\frac{\rho_2 I_2}{\langle AE \rangle_{-1}}, \quad (4.12)$$

$$m_{j41} = m_{j42} = m_{j43} = 0, \quad m_{j44} = (\rho_1 A_1 + \rho_2 A_2). \quad (4.13)$$

Further let

$$\mathbf{X}_j^T = [\bar{s}_j, \bar{\phi}_{1j}, \bar{\phi}_{2j}, \bar{v}_j]. \quad (4.14)$$

For the free vibrations of simply supported composite beams with weak shear connection we assume that

$$\bar{v}(z) = \sum_{j=1}^{\infty} \tilde{v}_j \sin j \frac{\pi}{L} z, \quad \bar{s}(z) = \sum_{j=1}^{\infty} \tilde{s}_j \cos j \frac{\pi}{L} z, \quad (4.15)$$

$$\bar{\phi}_1(z) = \sum_{j=1}^{\infty} \tilde{\phi}_{1j} \cos j \frac{\pi}{L} z, \quad \bar{\phi}_2(z) = \sum_{j=1}^{\infty} \tilde{\phi}_{2j} \cos j \frac{\pi}{L} z. \quad (4.16)$$

A comparison with equations (3.1) shows that the boundary conditions (3.2) are again satisfied. By repeating the line of thought resulting in equation (3.3) the eigenvalue problem for the free vibrations of simply supported two-layer beams with flexible shear connection can be formulated as

$$(\mathbf{C}_j + \omega^2 \mathbf{M}_j) \mathbf{X}_j = \mathbf{0}, \quad (j = 1, 2, \dots). \quad (4.17)$$

For each  $j$  we have three different natural frequencies. The smaller value of  $\omega_j^2$  corresponds to the bending deformation mode and the other two values of  $\omega_j^2$  correspond to the shear deformation modes. For the Euler-Bernoulli and Euler-Bernoulli-Rayleigh beams we have only bending deformation mode and the natural frequencies can be obtained – see [26] – from the following equation

$$\Omega_j^2 = \frac{\langle IE \rangle \left[ \left( \frac{j\pi}{L} \right)^4 + \left( \frac{j\Gamma}{L} \right)^2 \right] \left( \frac{j\pi}{L} \right)^2}{\left[ m + I_m \left( \frac{j\pi}{L} \right)^2 \right] \left[ \frac{k}{\langle AE \rangle_{-1}} + \left( \frac{j\pi}{L} \right)^2 \right]}, \quad (4.18)$$

where

$$\langle IE \rangle = E_i \int_{A_i} (y - y_i)^2 dA, \quad (i = 1, 2), \quad m = \rho_1 A_1 + \rho_2 A_2, \quad (4.19)$$

$$\Gamma^2 = \frac{k \{IE\}}{\langle AE \rangle_{-1} \langle IE \rangle}, \quad \{IE\} = \langle IE \rangle + (y_1 - y_2)^2 \langle AE \rangle_{-1}, \quad (4.20)$$

$$I_m = 0 \quad (\text{Euler-Bernoulli beam}), \quad (4.21)$$

$$I_m = \sum_{i=1}^2 \left[ \rho_i \int_{A_i} (y - y_i)^2 dA + y_i^2 \rho_i A_i \right] \quad (\text{Euler-Bernoulli-Rayleigh beam}) . \quad (4.22)$$

## 5. NUMERICAL EXAMPLES

**5.1. Simply supported beam loaded by uniform distributed force.** This example is taken from paper [18] by Schnabl et al. The simply supported beam, its cross-section and the applied load are shown in Figure 8. The following data are used  $h_1 = 0.2$  [m],  $h_2 = 0.3$  [m],  $b = 0.3$  [m],  $E_1 = 1.2 \times 10^{10}$  [Pa],  $E_2 = 1.2 \times 10^{10}$  [Pa],  $G_1 = 8 \times 10^8$  [Pa],  $G_2 = 1.2 \times 10^9$  [Pa],  $f = 5 \times 10^4$  [N/m],  $k = 2.43 \times 10^6$  [Pa],  $L = 2.5$  [m],  $\kappa_1 = \kappa_2 = 5/6$ . The functions  $v = v(z)$  and  $s = s(z)$  are shown in Figures 9 and 10. The functions  $\phi_1 = \phi_1(z)$  and  $\phi_2 = \phi_2(z)$  are also presented in graphical format – see Figure 11. A comparison of the deflection  $v(\frac{L}{2})$  and the slip  $s(0)$  with the results obtained by Schnabl et al. [18] is given in Table 2.

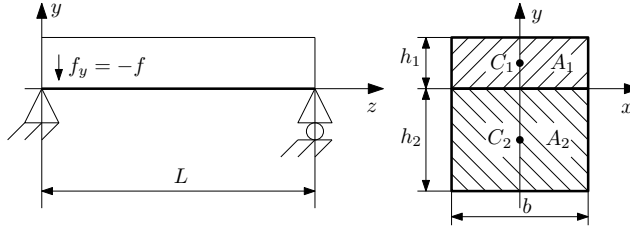


Figure 8. Simply supported composite beam with uniform load

Table 2. Comparison of deflection and slip

	paper [18] (FEM)	present paper
$v(L/2)$ [m]	-0.00271026	-0.0027082964
$s(0)$ [m]	-0.00077293	-0.0007713293

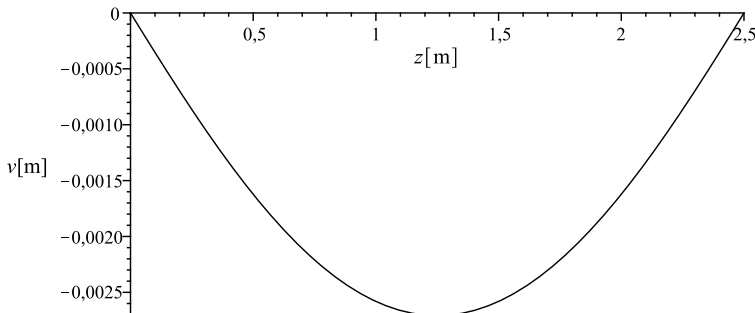


Figure 9. The plot of  $v(z)$

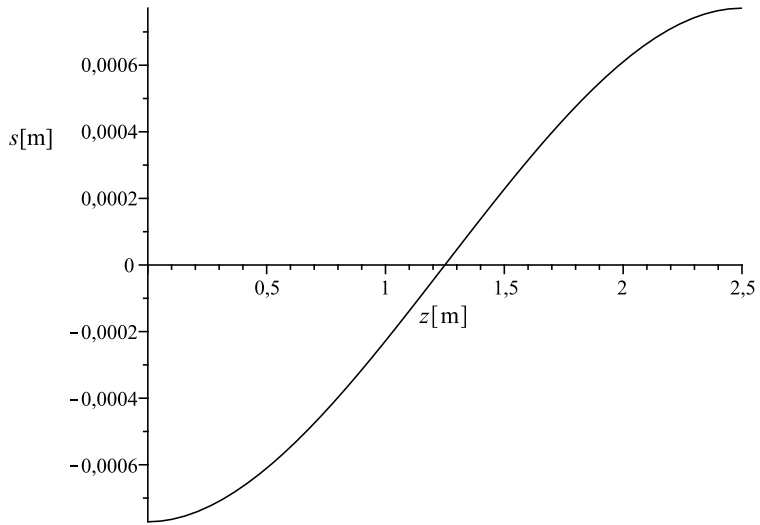


Figure 10. The plot of  $s(z)$

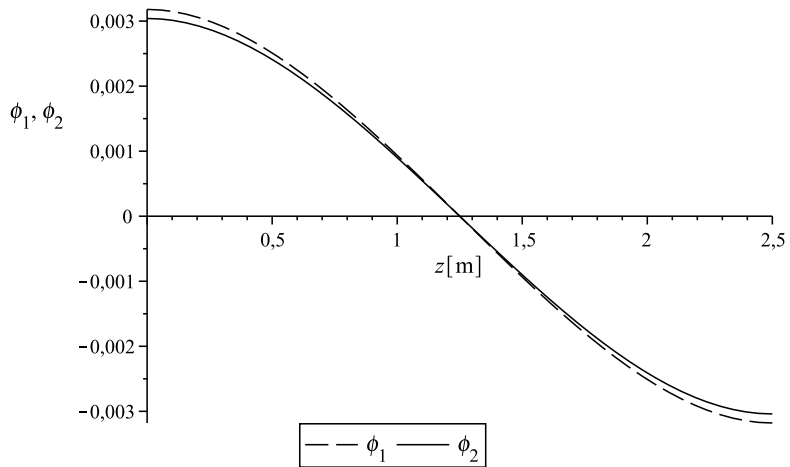


Figure 11. The plots of  $\phi_1(z)$  and  $\phi_2(z)$

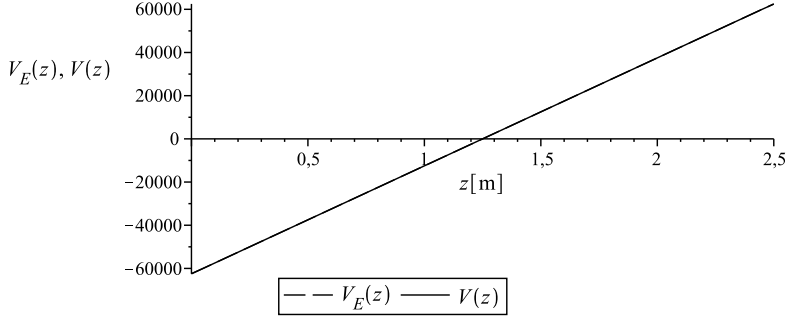


Figure 12. The shear force functions

For completeness Figure 12 shows the shear force

$$V_E(z) = f \left( z - \frac{L}{2} \right). \quad (5.1)$$

The shear force function  $V = V(z)$  is computed from the deflection and the cross-sectional rotation by utilizing equations (2.13), (2.27)<sub>1</sub>

$$V(z) = \kappa_1 G_1 A_1 \left( \frac{dv}{dz} + \phi_1 \right) + \kappa_2 G_2 A_2 \left( \frac{dv}{dz} + \phi_2 \right). \quad (5.2)$$

This function is also shown in Figure 12. The curves  $V_E(z)$  and  $V(z)$  coincide and this fact is evidence for the accuracy of the presented solutions.

**5.2. Natural frequencies of the free vibrations.** The data for this example are the same as those in Example 5.1. The densities should also be given:  $\rho_1 = 5000 \text{ [kg/m}^3\text{]}$ ,  $\rho_2 = 7000 \text{ [kg/m}^3\text{]}$ . The results we have obtained for the natural frequencies are listed in Table 3.

Table 3. Natural frequencies

j	Euler-Bernoulli	Euler-Bernoulli-Rayleigh	Timoshenko		
			Bending	Shear	Shear
1	155.290664	152.6194487	151.4225922	2338.54532	8394.253453
2	622.4043575	582.6170364	504.1041551	2901.662787	3724.202344
3	1400.929022	1220.369015	941.8521016	3635.127974	4202.393801
4	2490.863793	1991.111501	1414.604281	4451.987908	4779.655958
5	3892.20856	2836.587389	1902.205348	5291.520318	5444.286196

## 6. CONCLUSIONS

In this paper an analytical model has been developed to analyse the deformation of composite beams with weak shear connections. Timoshenko beam theory is used, assuming that the layers have different cross-sectional rotations. Analytical solutions

for deflection, slip and cross-sectional rotations are derived. The eigenfrequencies of free vibrations of a simply supported beam are also computed. The presented solutions are based on the representations of applied load, deflection, slip and cross-sectional rotations by Fourier series. The numerical results we have obtained are compared with a FEM solution and good agreement has been found.

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