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TIME RATES OF TENSORS IN CONTINUUM MECHANICS UNDER ARBITRARY TIME DEPENDENT TRANSFORMATIONS PART I. MATERIAL TIME RATES

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Abstract. The motion of a body and material time rates of tensors are investigated in three coordinate systems: in a fixed and a relative one – the latter is assumed to move arbitrarily with respect to the fixed coordinate system, i.e., it is deformable – and also in the convected coordinate system. Relations have been derived for the material time rates of arbitrary tensors in all three coordinate systems.

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1. Introduction

1.1. Components of tensors at a point of space can be transformed from one coordinate system into another by making use of the general transformation rules of tensors. If the coordinate systems move with respect to each other, one speaks about time dependent transformations.

If the motion of the coordinate systems relative to each other is arbitrary (one of the coordinate systems is deformed with respect to the other) then the transformation is also referred to as arbitrary, otherwise, i.e., for a rigid body motion as relative motion of the coordinate system, the transformation is an orthogonal one and in both cases time dependent.

From this point of view those tensors (including some time rate of tensors) which can be defined independently of the choice of coordinate systems moving arbitrarily with respect to each other, i.e., which are invariant under any arbitrary and time dependent transformations, will be referred to as physically (or materially) objective, or, for the sake of brevity, objective tensors or objective rates. (We remark that in the literature criteria of physical objectivity are valid mostly for orthogonal transformation only.)

Fulfillment of physical objectivity is a necessary (but not sufficient) condition for tensors (and time rates of tensors) fulfilling the criterion of material objectivity and disregards the issue of establishing constitutive equations.

1.2. The first objective time rate, the Jaumann stress rate [1] is related to a coordinate system rotating together with the spin tensor of continuum. Later on the Jaumann stress rate was also derived by other authors - for example by Fromm [2], Zaremba [3], Thomas [4], Noll [5] and Hill [6]. These authors have not referred to Jaumann's work. In the literature, however, the Jaumann's stress rate is generally accepted although Atluri [7] associates it with the names Zaremba - Jaumann - Noll.

A detailed description of some objective time rates is presented, among others, by Sedov [8], Prager [9], Naghdi and Wainwright [10], Atluri [7], Masur [11], Dubey [12], Szabó and Balla [13], Haupt and Tsakamakis [14].

There are some famous objective time rates beside the Jaumann stress rate mentioned above. Using convective coordinates objective time rates of tensors with contravariant or covariant components were set up by Oldroyd [15], Trusdell [16], Cotter and Rivlin [17] and with all possible subscripts and superscripts by Sedov [8] and Atluri [7]. Atluri also gave the objective time rates in a fixed coordinate system. The stress rate introduced by Trusdell [16] is that of the II.Piola - Kirchhoff stress tensor. The objective time rates defined by Green and Naghdi [18], Green and McInnis [19], Dienes [20] and Atluri [7] are all regarded in a coordinate system rotating together with the spin tensor of the rotation tensor obtained from the polar decomposition of the deformation gradient. The objective time rate of Sowerby and Chu [21] is related to a coordinate system rotating with the spin tensor taken in the principal axis of the strains in the present configuration.

The objective time rates in [1] an [15]-[21] are all that of the stress tensor and invariance under orthogonal transformation is considered as a criterion for material objectivity.

References [7]-[14] offer not only a survey on the objective time rates but also a sort of systematization. The latter is grounded on the fact that the objective time rates are defined with the aid of a certain movement of the continuum, usually by the mapping of the reference configuration onto the present configuration or by the transformation between the fixed and convected coordinate systems or by the motion of the principal axis of strains. In some cases invariance under orthogonal transformations is a requirement, in the remaining cases, however, it is not.

After a wide mathematical foundation the book [22] by Marsden and Hughes also deals with the physically objective time rates pointing out that "All so called objective rates of second order tensors are in fact Lie derivatives."

1.3. Part I. and Part II. of the present paper are aimed to introduce physically objective time rates on the basis of mechanical (kinematical) considerations only and makes the introduction of the concept independent of the possible motions of continuum.

To accomplish this goal the paper investigates tensors and time rates of tensors in coordinate systems moving arbitrarily with respect to each other or, which is the same thing, in coordinate systems which are deformable. We regard alternatively one of the two coordinate systems as fixed; the other is then in motion with respect to the fixed one.

1.4. Part I of the present paper investigates the motion of two distinct continua. One of the two continua is the coordinate system moving in the fixed coordinate system as a fictitious purely geometrical continuum. The other is the actual material continuum itself. At the same time the motion of the actual material continuum can be viewed both from the fixed coordinate system and from the one moving with respect to it.

In a particular case the convected coordinate system can also be regarded as a moving coordinate system or a fixed one (see, for example, Section 4). If this is the case, one should keep in mind that the continuum is at a relative rest in the convected coordinate system.

1.5. The next section investigates coordinate systems moving arbitrarily with respect to each other. Metric tensors, velocities, time derivatives of base vectors are also discussed.

Section 3 is devoted to the motion of a continuum in coordinate systems moving arbitrarily with respect to each other.

In Section 4 material time rates are defined in various coordinate systems including the fixed coordinate system, the coordinate system moving arbitrarily with respect to the fixed one and the convected coordinate system. The various time rates of the same tensor are related to each other and the corresponding relations are also presented.

1.6. We shall use both the indicial notations of tensors and the symbolic or direct notational system. The coordinate systems are arbitrary and curvilinear.

In accordance with the general rules of indicial notations - no matter whether the indices are minuscule or majuscule - indices range over the integers 1,2 and 3; summation over repeated indices is implied and the subscripts proceeded by a [comma] {semicolon} denotes [partial] {covariant} differentiation with respect to the corresponding variable. Underscore of indices suspends summation. δ^p_q stands for the Kronecker symbol.

As regards symbolic notations the dot product is denoted in the usual manner, i.e., by a dot placed between the factors, while no operation sign is employed to denote tensor products. If necessary, small asterisks are used to show where the indices stand, for example $\boldsymbol{A}^{*}{}_{*} = \widehat{a}^{k}{}_{l}\widehat{\mathbf{g}}_k\widehat{\mathbf{g}}^l$ in which $\widehat{\mathbf{g}}_k$ and $\widehat{\mathbf{g}}^l$ are the base vectors. (In the case of indicial notations it is obvious where the indices are.)

The transpose of a tensor is denoted by T. We shall utilize the fact that the covariant derivatives are defined independently of a coordinate system.

Time is common for all sets of variables. At the points of time $t_\circ$ and $t > t_\circ$ (otherwise t is arbitrary) the state of continuum is referred to as reference configuration and present configuration, respectively.

Further notations and notational conventions are presented at their first occurrence in the text.

2. Arbitrary motion of two coordinate systems with respect to each other

2.1. First let the coordinate system $\{x^p\}$ be the fixed one. The corresponding base vectors and the covariant metric tensor are

$$\mathbf{g}_p\left(x^1,x^2,x^3\right)=\frac{\partial\mathbf{r}}{\partial x^p},\qquad \mathbf{g}^q\left(x^1,x^2,x^3\right)\qquad\text{and}\qquad g_{pq}\left(x^1,x^2,x^3\right), \tag{2.1}$$

where $\mathbf{r}$ is the position vector of a point P in space.

Let the coordinate system moving arbitrarily with respect to the coordinate system $\{x^p\}$ be denoted by $\left\{\widehat{x}^k\right\}$. The motion of the coordinate system $\left\{\widehat{x}^k\right\}$ relative to the coordinate system $\{x^p\}$ can be given in the form

$$x^p=\,^{(\mathrm{G})}x^p\left(\widehat{x}^1,\widehat{x}^2,\widehat{x}^3;t\right),\qquad\text{where}\qquad ^{(\mathrm{G})}J=\det\frac{\partial^{(\mathrm{G})}x^p}{\partial\widehat{x}^k}\neq 0\,. \tag{2.2}$$

Here and in the sequel a subscript in paranthesis to the left of the variable is of informative nature.

The base vectors in the coordinate system $\left\{\widehat{x}^k\right\}$ are of the form

$$\widehat{\mathbf{g}}_k\left(\widehat{x}^1,\widehat{x}^2,\widehat{x}^3;t\right)=\frac{\partial\mathbf{r}}{\partial\widehat{x}^k}=\frac{\partial\mathbf{r}}{\partial x^p}\frac{\partial^{(\mathrm{G})}x^p}{\partial\widehat{x}^k}=\frac{\partial^{(\mathrm{G})}x^p}{\partial\widehat{x}^k}\mathbf{g}_p\,,\qquad \widehat{\mathbf{g}}^l\left(\widehat{x}^1,\widehat{x}^2,\widehat{x}^3;t\right)=\frac{\partial\widehat{x}^l}{\partial^{(\mathrm{G})}x^q}\mathbf{g}^q. \tag{2.3}$$

The transformation matrices also depend on time and the matrix $\frac{\partial\widehat{x}^l}{\partial^{(\mathrm{G})}x^p}$ is the inverse of the matrix $\frac{\partial^{(\mathrm{G})}x^p}{\partial\widehat{x}^k}$.

The covariant metric tensor in the coordinate system$\left\{\widehat{x}^k\right\}$ is

$$\widehat{g}_{kl}\left(\widehat{x}^1,\widehat{x}^2,\widehat{x}^3;t\right)=\frac{\partial^{(\mathrm{G})}x^p}{\partial\widehat{x}^k}\frac{\partial^{(\mathrm{G})}x^q}{\partial\widehat{x}^l}g_{pq},\qquad g_{pq}\left(x^1,x^2,x^3\right). \tag{2.4}$$

As can be seen with ease neither $\mathbf{g}_p$ nor g_{pq} depend on time for an observer being in the coordinate system $\{x^p\}$ while, on the contrary, both $\widehat{\mathbf{g}}_k$ and $\widehat{g}_{kl}$ are time dependent.

Components of a tensor $\boldsymbol{A}=a^p{}_q\mathbf{g}_p\mathbf{g}^q=\widehat{a}^k{}_l\widehat{\mathbf{g}}_k\widehat{\mathbf{g}}^l$ regarded in the coordinate systems $\{x^p\}$ and $\left\{\widehat{x}^k\right\}$ obey the transformation rule which follows from (2.3):

$$\widehat{a}^k{}_l=\frac{\partial\widehat{x}^k}{\partial^{(\mathrm{G})}x^p}\frac{\partial^{(\mathrm{G})}x^q}{\partial\widehat{x}^l}a^p{}_q. \tag{2.5}$$

2.2. Secondly let the coordinate system $\left\{\widehat{x}^k\right\}$ be the fixed one. In this case - for an observer in the coordinate system $\left\{\widehat{x}^k\right\}$ - neither the base vectors $\widehat{\mathbf{g}}_k$,$\widehat{\mathbf{g}}^l$ nor the corresponding metric tensor $\widehat{g}_{kl}$ depend on time:

$$\widehat{\mathbf{g}}_k\left(\widehat{x}^1,\widehat{x}^2,\widehat{x}^3\right)=\frac{\partial\mathbf{r}}{\partial\widehat{x}^k},\qquad \widehat{\mathbf{g}}^l\left(\widehat{x}^1,\widehat{x}^2,\widehat{x}^3\right),\qquad \widehat{g}_{kl}\left(\widehat{x}^1,\widehat{x}^2,\widehat{x}^3\right). \tag{2.6}$$

For the motion of the coordinate system $\{x^p\}$ relative to the coordinate system $\left\{\widehat{x}^k\right\}$ we can write

$$\widehat{x}^k=\,^{(\mathrm{F})}\widehat{x}^k\left(x^1,x^2,x^3;t\right),\qquad ^{(\mathrm{F})}J=\det\frac{\partial^{(\mathrm{F})}\widehat{x}^k}{\partial x^p}\neq 0\,. \tag{2.7}$$

In this case the base vectors in the coordinate system $\{x^p\}$ are

$$\mathbf{g}_p\left(x^1, x^2, x^3; t\right) = \frac{\partial \mathbf{r}}{\partial x^p} = \frac{\partial \mathbf{r}}{\partial \widehat{x}^k}\frac{\partial^{(\mathrm{F})}\widehat{x}^k}{\partial x^p} = \frac{\partial^{(\mathrm{F})}\widehat{x}^k}{\partial x^p}\widehat{\mathbf{g}}_k, \qquad \mathbf{g}^q\left(x^1, x^2, x^3; t\right) = \frac{\partial x^q}{\partial^{(\mathrm{F})}\widehat{x}^l}\widehat{\mathbf{g}}^l\,. \tag{2.8}$$

The covariant metric tensor in the coordinate system $\{x^p\}$ takes the form

$$g_{pq}\left(x^1, x^2, x^3; t\right) = \frac{\partial^{(\mathrm{F})}\widehat{x}^k}{\partial x^p}\frac{\partial^{(\mathrm{F})}\widehat{x}^l}{\partial x^q}\widehat{g}_{kl}, \qquad \widehat{g}_{kl}\left(\widehat{x}^1, \widehat{x}^2, \widehat{x}^3\right)\,. \tag{2.9}$$

The transformation matrices also depend on time and the matrix $\frac{\partial x^p}{\partial^{(\mathrm{F})}\widehat{x}^k}$ is the inverse of the matrix $\frac{\partial^{(\mathrm{F})}\widehat{x}^k}{\partial x^p}$.

The apparent contradiction between the formulae (2.4) and (2.9) giving the metric tensors follows from the fact that time dependence of tensor components depends on which coordinate system is regarded as a fixed one. If the coordinate system $\{x^p\}$ is the fixed one, g_{pq} is independent of time, but $\widehat{g}_{kl}$ is time dependent and, on the contrary, if $\left\{\widehat{x}^k\right\}$ is the fixed coordinate system $\widehat{g}_{kl}$ is independent of time while g_{pq} is a function of time.

2.3. In the sequel - unless the opposite is stated - *we shall always assume that the coordinate system $\{x^p\}$ is a fixed one while the coordinate system $\left\{\widehat{x}^k\right\}$, which will be referred to as grid, is the moving one.* [Use of the letter "*F*" (fixed) and "*G*" (grid) for the motions (2.7) and (2.2) implies this convention tacitly.]

This general convention means no limitation either on the arbitrariness of the motion of coordinate systems relative to each other or on the general validity of the conclusions we hope to come to.

In what follows

- the motion of a material continuum with respect to the fixed coordinate system $\{x^p\}$ will be referred to simply as *motion* or *absolute motion*,
- the motion of a material continuum with respect to the coordinate system $\left\{\widehat{x}^k\right\}$, i.e., to the grid will be referred to as *relative motion*
- and the motion of the coordinate system $\left\{\widehat{x}^k\right\}$, i.e., that of the grid with respect to the fixed coordinate system $\{x^p\}$ will be called *the motion of grid* or *grid motion.*

2.4. Velocity of a point with coordinates $\widehat{x}^k$ of the grid with respect to $\{x^p\}$ follows from the grid motion (2.2):

$$^{(\mathrm{Gx})}\mathbf{v} = \left.\frac{\partial \mathbf{r}}{\partial t}\right|_{(\widehat{x})} = \left.\frac{\partial \mathbf{r}}{\partial x^p}\frac{\partial^{(\mathrm{G})}x^p}{\partial t}\right|_{(\widehat{x})} = {}^{(\mathrm{Gx})}v^p\mathbf{g}_p = {}^{(\mathrm{Gx})}\widehat{v}^k\widehat{\mathbf{g}}_k, \tag{2.10}$$

$$\text{where} \quad {}^{(\mathrm{Gx})}v^p = \left.\frac{\partial^{(\mathrm{G})}x^p}{\partial t}\right|_{(\widehat{x})} \quad \text{and} \quad {}^{(\mathrm{Gx})}\widehat{v}^k = \frac{\partial \widehat{x}^k}{\partial^{(\mathrm{G})}x^p}\,{}^{(\mathrm{Gx})}v^p \tag{2.11}$$

are the components of the velocity vector $^{(\mathrm{Gx})}\mathbf{v}$ in the coordinate systems $\{x^p\}$ and $\left\{\widehat{x}^k\right\}$. Here and in the sequel a subscript placed to the right of a vertical line – the latter is a right delimiter – refers to the fact that the corresponding coordinates are constants when one determines a time derivative.

With regard to all that has been said about the coordinate systems $\{x^p\}$ and $\{\widehat{x}^k\}$ it is obvious that their roles are interchangeable. Comparison of (2.2) and (2.7) yields an identity

$$x^p = {}^{(\mathrm{G})}x^p\left[{}^{(\mathrm{F})}\widehat{x}^1\left(x^1, x^2, x^3; t\right), \widehat{x}^2(\cdots), \widehat{x}^3(\cdots); t\right] = x^p$$

which is valid at any point of the grid and from which, taking into account that the points $\{x^p\}$ do not move at all, a further identity follows:

$$\left.\frac{\partial x^p}{\partial t}\right|_{(x)} = 0 = \left.\frac{\partial^{(\mathrm{G})}x^q}{\partial t}\right|_{(\widehat{x})} + \frac{\partial^{(\mathrm{G})}x^p}{\partial \widehat{x}^k}\left.\frac{\partial^{(\mathrm{F})}\widehat{x}^k}{\partial t}\right|_{(x)} \tag{2.12}$$

$$\text{where} \quad \left.\frac{\partial^{(\mathrm{F})}\widehat{x}^k}{\partial t}\right|_{(x)} = {}^{(\mathrm{F}\widehat{\mathrm{x}})}\widehat{v}^k. \tag{2.13}$$

Let x^p be a point of the coordinate system $\{x^p\}$. After substituting (2.13) and (2.11) into (2.12) we obtain the velocity ${}^{(\mathrm{F}\widehat{\mathrm{x}})}\mathbf{v}$ of the point x^p with respect to the grid – that is to the coordinate system $\{\widehat{x}^k\}$:

$${}^{(\mathrm{F}\widehat{\mathrm{x}})}\widehat{v}^k = -\frac{\partial\widehat{x}^k}{\partial^{(\mathrm{G})}x^p}\,{}^{(\mathrm{Gx})}v^p = -\,{}^{(\mathrm{Gx})}\widehat{v}^k, \quad \text{i.e.,} \quad {}^{(\mathrm{F}\widehat{\mathrm{x}})}\mathbf{v} = -\,{}^{(\mathrm{Gx})}\mathbf{v}\,. \tag{2.14}$$

2.5. From the velocity vector field of the grid ${}^{(\mathrm{Gx})}\mathbf{v}$ we can obtain, in the usual manner, the velocity gradient, the strain rate tensor and the spin tensor for the grid motion:

$${}^{(\mathrm{Gx})}\boldsymbol{L} = {}^{(\mathrm{Gx})}l^p{}_q\mathbf{g}_p\mathbf{g}^q = {}^{(\mathrm{Gx})}\mathbf{v}\nabla, \quad {}^{(\mathrm{Gx})}l^p{}_q = {}^{(\mathrm{Gx})}v^p{}_{;q}, \tag{2.15}$$

$${}^{(\mathrm{Gx})}\boldsymbol{D} = {}^{(\mathrm{Gx})}d^p{}_q\mathbf{g}_p\mathbf{g}^q = \frac{1}{2}\left({}^{(\mathrm{Gx})}\boldsymbol{L} + {}^{(\mathrm{Gx})}\boldsymbol{L}^{\mathrm{T}}\right), \quad {}^{(\mathrm{Gx})}d^p{}_q = \frac{1}{2}\left({}^{(\mathrm{Gx})}l^p{}_q + {}^{(\mathrm{Gx})}l_q{}^p\right), \tag{2.16}$$

$${}^{(\mathrm{Gx})}\boldsymbol{W} = {}^{(\mathrm{Gx})}w^p{}_q\mathbf{g}_p\mathbf{g}^q = \frac{1}{2}\left({}^{(\mathrm{Gx})}\boldsymbol{L} - {}^{(\mathrm{Gx})}\boldsymbol{L}^{\mathrm{T}}\right), \quad {}^{(\mathrm{Gx})}w^p{}_q = \frac{1}{2}\left({}^{(\mathrm{Gx})}l^p{}_q - {}^{(\mathrm{Gx})}l_q{}^p\right). \tag{2.17}$$

By making use of the transformation rule (2.5) we can readily obtain the components ${}^{(\mathrm{Gx})}\widehat{l}^p{}_q$, ${}^{(\mathrm{Gx})}\widehat{d}^p{}_q$, ${}^{(\mathrm{Gx})}\widehat{w}^p{}_q$ as well, i.e., the components of the previous tensors in the coordinate system $\{\widehat{x}^k\}$.

Recalling the definition of ${}^{(\mathrm{Gx})}\boldsymbol{L}$ – see equation (2.15) – , we have

$$\left.\frac{\partial}{\partial t}(\mathrm{d}\mathbf{r})\right|_{(\widehat{x})} = \mathrm{d}\left({}^{(\mathrm{Gx})}\mathbf{v}\right) = {}^{(\mathrm{Gx})}\boldsymbol{L}\cdot\mathrm{d}\mathbf{r}\,. \tag{2.18}$$

2.6. In accordance with (2.18) it also holds that

$$\left.{}^{(x)}\frac{\partial\widehat{\mathbf{g}}_k}{\partial t}\right|_{(\widehat{x})} = {}^{(\mathrm{Gx})}\boldsymbol{L}\cdot\widehat{\mathbf{g}}_k, \qquad \left.{}^{(x)}\frac{\partial\widehat{\mathbf{g}}^l}{\partial t}\right|_{(\widehat{x})} = -\,\widehat{\mathbf{g}}^l\cdot{}^{(\mathrm{Gx})}\boldsymbol{L}\,. \tag{2.19}$$

3. Motion of a body in the absolute coordinate system and the grid

3.1. Let $\{X^K\}$ be the convected coordinate system. Further let the motion of the body with respect to the coordinate system $\{\widehat{x}^k\}$ be

$$\widehat{x}^k = {}^{(\mathrm{B})}\widehat{x}^k\left(X^1, X^2, X^3; t\right), \qquad {}^{(\mathrm{B})}J = \det \frac{\partial^{(\mathrm{B})}\widehat{x}^k}{\partial X^K} \neq 0\,. \tag{3.1}$$

This motion is the relative motion of the body. By

$$x^p = {}^{(\mathrm{B})}x^p\left(X^1, X^2, X^3; t\right), \qquad J = \det \frac{\partial^{(\mathrm{B})}x^p}{\partial X^K} \neq 0 \tag{3.2}$$

we denote the motion of the body with respect to $\{x^p\}$ – absolute motion of the body.

With the relative motion of the body it follows that

$$x^p = {}^{(\mathrm{B})}x^p\left(X^1, X^2, X^3; t\right) = {}^{(\mathrm{G})}x^p\left[{}^{(\mathrm{B})}\widehat{x}^1\left(X^1, X^2, X^3; t\right), {}^{(\mathrm{B})}\widehat{x}^2(\cdots), {}^{(\mathrm{B})}\widehat{x}^3(\cdots); t\right] \tag{3.3}$$

$$\text{and} \quad J = \det \frac{\partial^{(\mathrm{B})}x^p}{\partial X^K} = \det \frac{\partial^{(\mathrm{G})}x^p}{\partial \widehat{x}^k}\frac{\partial^{(\mathrm{B})}\widehat{x}^k}{\partial X^K} = {}^{(\mathrm{G})}J\;{}^{(\mathrm{B})}J \neq 0.$$

3.2. The base vectors of the coordinate system $\{X^K\}$ assume the form

$$\widehat{\mathbf{G}}_K = \frac{\partial \mathbf{r}}{\partial X^K} = \frac{\partial \mathbf{r}}{\partial x^p}\frac{\partial^{(\mathrm{B})}x^p}{\partial X^K} = \frac{\partial^{(\mathrm{B})}x^p}{\partial X^K}\mathbf{g}_p, \qquad \widehat{\mathbf{G}}^L = \frac{\partial X^L}{\partial^{(\mathrm{B})}x^q}\mathbf{g}^q, \tag{3.4}$$

from which using (2.3) and (3.3) we obtain

$$\widehat{\mathbf{G}}_K = \frac{\partial^{(\mathrm{B})}x^p}{\partial X^K}\frac{\partial \widehat{x}^k}{\partial^{(\mathrm{G})}x^p}\widehat{\mathbf{g}}_k = \frac{\partial^{(\mathrm{G})}x^p}{\partial \widehat{x}^l}\frac{\partial^{(\mathrm{B})}\widehat{x}^l}{\partial X^K}\frac{\partial \widehat{x}^k}{\partial^{(\mathrm{G})}x^p}\widehat{\mathbf{g}}_k = \frac{\partial^{(\mathrm{B})}\widehat{x}^k}{\partial X^K}\widehat{\mathbf{g}}_k, \qquad \widehat{\mathbf{G}}^L = \frac{\partial X^L}{\partial^{(\mathrm{B})}\widehat{x}^l}\widehat{\mathbf{g}}^l. \tag{3.5}$$

The transformation matrices are again time dependent and the matrices $\frac{\partial X^L}{\partial^{(\mathrm{B})}x^q}$ and $\frac{\partial X^L}{\partial^{(\mathrm{B})}\widehat{x}^l}$ are respectively inverses of the matrices $\frac{\partial^{(\mathrm{B})}x^p}{\partial X^K}$ and $\frac{\partial^{(\mathrm{B})}\widehat{x}^k}{\partial X^K}$.

Using (3.4) and (3.5) for the covariant metric tensor of the coordinate system $\{X^K\}$ we can write

$$\widehat{G}_{KL} = \frac{\partial^{(\mathrm{B})}x^p}{\partial X^K}\frac{\partial^{(\mathrm{B})}x^q}{\partial X^L}g_{pq} = \frac{\partial^{(\mathrm{B})}\widehat{x}^k}{\partial X^K}\frac{\partial^{(\mathrm{B})}\widehat{x}^l}{\partial X^L}\widehat{g}_{kl}. \tag{3.6}$$

In view of (3.4) and (3.5) the components of a tensor $\boldsymbol{A} = a^p{}_q\mathbf{g}_p\mathbf{g}^q = \widehat{a}^k{}_l\widehat{\mathbf{g}}_k\widehat{\mathbf{g}}^l = \widehat{a}^K{}_L\widehat{\mathbf{G}}_K\widehat{\mathbf{G}}^L$, which is regarded in the fixed coordinate system $\{x^p\}$ and the coordinate systems $\{\widehat{x}^k\}$ and $\{X^K\}$ each moving with respect to the fixed one, should follow the transformation rules

$$\widehat{a}^K{}_L = \frac{\partial X^K}{\partial^{(\mathrm{B})}x^p}\frac{\partial^{(\mathrm{B})}x^q}{\partial X^L}a^p{}_q = \frac{\partial X^K}{\partial^{(\mathrm{B})}\widehat{x}^K}\frac{\partial^{(\mathrm{B})}\widehat{x}^l}{\partial X^L}\widehat{a}^k{}_l. \tag{3.7}$$

Without entering into further details, we mention that for the case when the convected coordinate system $\{X^K\}$ is chosen as a fixed one

$$\widehat{\mathbf{G}}_K\left(X^1, X^2, X^3\right) = \frac{\partial \mathbf{r}}{\partial X^K}, \qquad \widehat{\mathbf{G}}^L\left(X^1, X^2, X^3\right). \tag{3.8}$$

are the base vectors observed from the coordinate system itself.

3.3 The quantities we have defined so far are associated with the current configuration and are regarded at the spatial point P. The quantities that are regarded at the points of the reference configuration will be denoted by barred letters (for example $\overline{\mathbf{G}}_K$ or $\mathrm{d}\overline{\mathbf{r}}$).

Motion (3.2) is a mapping of the reference configuration onto the current configuration. The deformation gradient

$$\boldsymbol{F} = \frac{\partial^{(\mathrm{B})} x^p}{\partial X^L}\mathbf{g}_p\overline{\mathbf{G}}^L \quad \text{and its inverse} \quad \boldsymbol{F}^{-1} = \frac{\partial X^K}{\partial^{(\mathrm{B})} x^q}\overline{\mathbf{G}}_K\mathbf{g}^q \tag{3.9}$$

represent a linear mapping and remapping between the line elements $\mathrm{d}\overline{\mathbf{r}} = \mathrm{d}X^K\overline{\mathbf{G}}_K$ and $\mathrm{d}\mathbf{r} = \mathrm{d}x^p\mathbf{g}_p$ regarded, respectively, in the reference and current configurations:

$$\mathrm{d}\mathbf{r} = \boldsymbol{F}\cdot\mathrm{d}\overline{\mathbf{r}}, \qquad \mathrm{d}\overline{\mathbf{r}} = \boldsymbol{F}^{-1}\cdot\mathrm{d}\mathbf{r}. \tag{3.10}$$

With the line elements $\mathrm{d}\overline{\mathbf{r}} = \mathrm{d}\overline{s}\,\overline{\mathbf{e}}$ and $\mathrm{d}\mathbf{r} = \mathrm{d}s\,\mathbf{e}$, in which $\overline{\mathbf{e}}$ and $\mathbf{e}$ are unit vectors in the reference and current configurations, one can define stretches in the directions $\overline{\mathbf{e}}$ and $\mathbf{e}$:

$$\lambda_e = \frac{\mathrm{d}s}{\mathrm{d}\overline{s}}, \qquad \lambda_e^2 = \overline{\mathbf{e}}\cdot\boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{F}\cdot\overline{\mathbf{e}} = \frac{1}{\mathbf{e}\cdot\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\cdot\boldsymbol{F}^{-1}\cdot\mathbf{e}}. \tag{3.11}$$

3.4. By using the polar decomposition theorem, the deformation gradient $\boldsymbol{F}$ ($\det\boldsymbol{F} = J \neq 0$) can be decomposed into the dot product of the rotation tensor

$$\boldsymbol{R} = R^p{}_K\mathbf{g}_p\overline{\mathbf{G}}^K, \qquad \boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}} \tag{3.12}$$

with the right and left stretch tensors

$$\overline{\boldsymbol{U}} = \overline{U}_L^K\overline{\mathbf{G}}_K\overline{\mathbf{G}}^L, \qquad \boldsymbol{V} = V_q^p\mathbf{g}_p\mathbf{g}^q \tag{3.13}$$

in a unique fashion:

$$\boldsymbol{F} = \boldsymbol{R}\cdot\overline{\boldsymbol{U}} = \boldsymbol{V}\cdot\boldsymbol{R}. \tag{3.14}$$

Here the tensors $\overline{\boldsymbol{U}}$ and $\boldsymbol{V}$ are defined in the reference and current configurations and are both positive definite and symmetric tensors while the tensor $\boldsymbol{R}$ is orthogonal.

Let $\overline{\mathbf{n}}_p$ and $\mathbf{n}_q$ be orthonormal eigenvectors directed along the principal axes of the stretch tensors $\overline{\boldsymbol{U}}$ and $\boldsymbol{V}$, respectively. The coordinate systems constituted by the principal axes of the right and left stretch tensors are denoted by $\{\overline{\nu}^p\}$ and $\{\nu^p\}$. It can be shown that

$$\mathbf{n}_p = \boldsymbol{R}\cdot\overline{\mathbf{n}}_p, \qquad \overline{\mathbf{n}}_p = \boldsymbol{R}^{\mathrm{T}}\cdot\mathbf{n}_p \tag{3.15}$$

The tensors $\overline{\boldsymbol{U}}$ and $\boldsymbol{V}$ have the same eigenvalues (denoted by λ_p). In the coordinate systems $\{\overline{\nu}^p\}$ and $\{\nu^p\}$ we have

$$\overline{\boldsymbol{U}} = \lambda_{\underline{p}}\delta^p_q\overline{\mathbf{n}}_p\overline{\mathbf{n}}^q, \quad \overline{\boldsymbol{U}}^{-1} = \frac{1}{\lambda_{\underline{p}}}\delta^p_q\overline{\mathbf{n}}_p\overline{\mathbf{n}}^q, \qquad \boldsymbol{V} = \lambda_{\underline{p}}\delta^p_q\mathbf{n}_p\mathbf{n}^q, \quad \boldsymbol{V}^{-1} = \frac{1}{\lambda_{\underline{p}}}\delta^p_q\mathbf{n}_p\mathbf{n}^q. \tag{3.16}$$

In addition we define the *Hencky strain tensor* in the coordinate system $\{\nu^p\}$

$$\ln\boldsymbol{V} = \ln\lambda_{\underline{p}}\delta^p_q\mathbf{n}_p\mathbf{n}^q. \tag{3.17}$$

3.5. The velocity vector of the moving continuum at the point with coordinates X^K observed from the coordinate system $\{x^p\}$ can be obtained from the motion (3.2):

$$^{(\mathrm{x})}\mathbf{v} = \left.\frac{\partial\mathbf{r}}{\partial t}\right|_{(X)} = \frac{\partial\mathbf{r}}{\partial x^p}\left.\frac{\partial^{(\mathrm{B})}x^p}{\partial t}\right|_{(X)} = {}^{(\mathrm{x})}v^p\mathbf{g}_p, \qquad {}^{(\mathrm{x})}v^p = \left.\frac{\partial^{(\mathrm{B})}x^p}{\partial t}\right|_{(X)}. \tag{3.18}$$

From the velocity vector field $^{(\mathrm{x})}\mathbf{v}$ of the moving continuum we can derive the velocity gradient, the strain rate tensor and the spin tensor:

$$^{(\mathrm{x})}\boldsymbol{L} = {}^{(\mathrm{x})}l^p{}_q\mathbf{g}_p\mathbf{g}^q = {}^{(\mathrm{x})}\mathbf{v}\nabla, \quad {}^{(\mathrm{x})}l^p{}_q = {}^{(\mathrm{x})}v^p{}_{;q}, \tag{3.19}$$

$$^{(\mathrm{x})}\boldsymbol{D} = {}^{(\mathrm{x})}d^p{}_q\mathbf{g}_p\mathbf{g}^q = \frac{1}{2}\left({}^{(\mathrm{x})}\boldsymbol{L} + {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}\right), \quad {}^{(\mathrm{x})}d^p{}_q = \frac{1}{2}\left({}^{(\mathrm{x})}l^p{}_q + {}^{(\mathrm{x})}l_q{}^p\right), \tag{3.20}$$

$$^{(\mathrm{x})}\boldsymbol{W} = {}^{(\mathrm{x})}w^p{}_q\mathbf{g}_p\mathbf{g}^q = \frac{1}{2}\left({}^{(\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}\right), \quad {}^{(\mathrm{x})}w^p{}_q = \frac{1}{2}\left({}^{(\mathrm{x})}l^p{}_q - {}^{(\mathrm{x})}l_q{}^p\right). \tag{3.21}$$

The velocity vector at the point with coordinates X^K of the continuum being now observed from the coordinate system $\{\widehat{x}^k\}$ (from the grid), i.e., the relative velocity of the continuum follows from the the relative motion (3.1):

$$^{(\widehat{\mathrm{x}})}\mathbf{v} = \frac{\partial\mathbf{r}}{\partial\widehat{x}^k}\left.\frac{\partial^{(\mathrm{B})}\widehat{x}^k}{\partial t}\right|_{(X)} = {}^{(\widehat{\mathrm{x}})}\widehat{v}^k\widehat{\mathbf{g}}_k, \qquad {}^{(\widehat{\mathrm{x}})}\widehat{v}^k = \left.\frac{\partial\widehat{x}^k}{\partial t}\right|_{(X)}. \tag{3.22}$$

From the relative velocity vector field of the continuum we can obtain the relative velocity gradient, the relative strain rate tensor and the relative spin tensor:

$$^{(\widehat{\mathrm{x}})}\boldsymbol{L} = {}^{(\widehat{\mathrm{x}})}\widehat{l}^k{}_l\widehat{\mathbf{g}}_k\widehat{\mathbf{g}}^l = \frac{\partial}{\partial\widehat{x}^l}\left({}^{(\widehat{\mathrm{x}})}\mathbf{v}\right)\widehat{\mathbf{g}}^l, \quad {}^{(\widehat{\mathrm{x}})}\widehat{l}^k{}_l = {}^{(\widehat{\mathrm{x}})}\widehat{v}^k{}_{;l}, \tag{3.23}$$

$$^{(\widehat{\mathrm{x}})}\boldsymbol{D} = {}^{(\widehat{\mathrm{x}})}\widehat{d}^k{}_l\widehat{\mathbf{g}}_k\widehat{\mathbf{g}}^l = \frac{1}{2}\left({}^{(\widehat{\mathrm{x}})}\boldsymbol{L} + {}^{(\widehat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}\right), \quad \widehat{d}^k{}_l = \frac{1}{2}\left({}^{(\widehat{\mathrm{x}})}\widehat{l}^k{}_l + {}^{(\widehat{\mathrm{x}})}\widehat{l}_l{}^k\right), \tag{3.24}$$

$$^{(\widehat{\mathrm{x}})}\boldsymbol{W} = {}^{(\widehat{\mathrm{x}})}\widehat{w}^k{}_l\widehat{\mathbf{g}}_k\widehat{\mathbf{g}}^l = \frac{1}{2}\left({}^{(\widehat{\mathrm{x}})}\boldsymbol{L} - {}^{(\widehat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}\right), \quad {}^{(\widehat{\mathrm{x}})}\widehat{w}^k{}_l = \frac{1}{2}\left({}^{(\widehat{\mathrm{x}})}\widehat{l}^k{}_l - {}^{(\widehat{\mathrm{x}})}\widehat{l}_l{}^k\right). \tag{3.25}$$

It follows from the nature of things that the velocity at the point X^K of the continuum with respect to the convected coordinate system $\{X^K\}$ vanishes: $^{(\mathrm{X})}\mathbf{v} = \mathbf{0}$, $^{(\mathrm{X})}\widehat{v}^K = 0$.

Similarly, it can be checked with ease that $^{(\mathrm{X})}\boldsymbol{L} = {}^{(\mathrm{X})}\boldsymbol{D} = {}^{(\mathrm{X})}\boldsymbol{W} = \boldsymbol{0}$.

3.6. Velocities of the points of continuum defined in the coordinate systems $\{x^p\}$ and $\{\widehat{x}^k\}$ can be related to each other by using the motion (2.2) of the grid and the relative motion (3.1) of continuum:

$$ {}^{(\mathrm{x})}\mathbf{v} = \left.\frac{\partial \mathbf{r}}{\partial t}\right|_{(X)} = \frac{\partial \mathbf{r}}{\partial x^p}\left(\left.\frac{\partial^{(\mathrm{G})} x^p}{\partial t}\right|_{(\widehat{x})} + \frac{\partial^{(\mathrm{G})} x^p}{\partial \widehat{x}^k}\left.\frac{\partial^{(\mathrm{B})} \widehat{x}^k}{\partial t}\right|_{(X)} \right). $$

Substituting (2.10) and (3.22) we arrive at the result

$$ {}^{(\mathrm{x})}v^p = {}^{(\mathrm{Gx})}v^p + \frac{\partial^{(\mathrm{G})} x^p}{\partial \widehat{x}^k}\, {}^{(\widehat{\mathrm{x}})}\widehat{v}^k = {}^{(\mathrm{Gx})}v^p + {}^{(\widehat{\mathrm{x}})}v^p \quad \text{or} \quad {}^{(\mathrm{x})}\mathbf{v} = {}^{(\mathrm{Gx})}\mathbf{v} + {}^{(\widehat{\mathrm{x}})}\mathbf{v}. \tag{3.26} $$

From equation (3.26), which relates the various velocities to each other, taking equations (2.15)-(2.17), (3.19)-(3.21) and (3.23)-(3.25) into account, we can readily establish further equations for the velocity gradients, the strain rate tensors and the spin tensors:

$$ {}^{(\mathrm{x})}\boldsymbol{L} = {}^{(\mathrm{Gx})}\boldsymbol{L} + {}^{(\widehat{\mathrm{x}})}\boldsymbol{L} \quad {}^{(\mathrm{x})}l^p{}_q = {}^{(\mathrm{Gx})}l^p{}_q + {}^{(\widehat{\mathrm{x}})}l^p{}_q, \tag{3.27} $$

$$ {}^{(\mathrm{x})}\boldsymbol{D} = {}^{(\mathrm{Gx})}\boldsymbol{D} + {}^{(\widehat{\mathrm{x}})}\boldsymbol{D}, \quad {}^{(\mathrm{x})}d^p{}_q = {}^{(\mathrm{Gx})}d^p{}_q + {}^{(\widehat{\mathrm{x}})}d^p{}_q, \tag{3.28} $$

$$ {}^{(\mathrm{x})}\boldsymbol{W} = {}^{(\mathrm{Gx})}\boldsymbol{W} + {}^{(\widehat{\mathrm{x}})}\boldsymbol{W}, \quad {}^{(\mathrm{x})}w^p{}_q = {}^{(\mathrm{Gx})}w^p{}_q + {}^{(\widehat{\mathrm{x}})}w^p{}_q. \tag{3.29} $$

The component forms of equations (3.27)-(3.29) can be written not only in the coordinate system $\{x^p\}$, but also in the coordinate system $\{\widehat{x}^k\}$ and $\{X^K\}$.

3.7. On the analogy of equation (2.19) we can obtain the time derivatives [measured in the coordinate system $\{x^p\}$] of the base vectors $\widehat{\mathbf{G}}_K$ and $\widehat{\mathbf{G}}^L$:

$$ \left.\overset{(\mathrm{x})}{\vdots}\frac{\partial \widehat{\mathbf{G}}_K}{\partial t}\right|_{(X)} = {}^{(\mathrm{x})}\boldsymbol{L}\cdot\widehat{\mathbf{G}}_K, \qquad \left.\overset{(\mathrm{x})}{\vdots}\frac{\partial \widehat{\mathbf{G}}^L}{\partial t}\right|_{(X)} = -\,\widehat{\mathbf{G}}^L\cdot\, {}^{(\mathrm{x})}\boldsymbol{L}. \tag{3.30} $$

Similarly, for the time derivatives of the base vectors $\widehat{\mathbf{G}}_K$ and $\widehat{\mathbf{G}}^L$[measured in the coordinate system $\{\widehat{x}^k\}$] we have:

$$ \left.\overset{(\widehat{\mathrm{x}})}{\vdots}\frac{\partial \widehat{\mathbf{G}}_K}{\partial t}\right|_{(X)} = {}^{(\widehat{\mathrm{x}})}\boldsymbol{L}\cdot\widehat{\mathbf{G}}_K, \qquad \left.\overset{(\widehat{\mathrm{x}})}{\vdots}\frac{\partial \widehat{\mathbf{G}}^L}{\partial t}\right|_{(X)} = -\,\widehat{\mathbf{G}}^L\cdot\, {}^{(\widehat{\mathrm{x}})}\boldsymbol{L}. \tag{3.31} $$

4. Material time rates of tensors

4.1. First, we shall separately define material time derivatives in the coordinate systems $\{x^p\}$, $\{\widehat{x}^k\}$ and $\{X^K\}$. Then we are seeking relations between the material time derivatives so introduced. Special care will be given to the metric tensor.

Consider the tensor fields

$$\begin{aligned} \boldsymbol{A} &= a_{pq}\mathbf{g}^p\mathbf{g}^q, & \boldsymbol{B} &= \widehat{b}_{kl}\widehat{\mathbf{g}}^k\mathbf{g}^l, & \boldsymbol{C} &= \widehat{c}_{KL}\widehat{\mathbf{G}}^K\widehat{\mathbf{G}}^L, \\ & a_{pq}\left(x^1,x^2,x^3;t\right), & & \mathbf{g}^p\left(x^1,x^2,x^3\right), \\ & \widehat{b}_{kl}\left(\widehat{x}^1,\widehat{x}^2,\widehat{x}^3;t\right), & & \widehat{\mathbf{g}}^k\left(\widehat{x}^1,\widehat{x}^2,\widehat{x}^3\right), \\ & \widehat{c}_{KL}\left(X^1,X^2,X^3;t\right), & & \widehat{\mathbf{G}}^K\left(X^1,X^2,X^3\right). \end{aligned}$$

written in the various coordinate systems as if they were fixed coordinate systems. By material time derivatives defined in the coordinate systems introduced, and for the tensors listed above we mean the time rate of change of the given tensor with respect to the coordinate system in which the tensor is defined and taken at the material point identified by the convected coordinates $\left\{X^K\right\}$.

Taking the possibilities one by one

- if $\{x^p\}$ is the defining coordinate system in which the continuum moves according to equation (3.2) and with the velocity ${}^{(\mathrm{x})}\mathbf{v}$ given by equation (3.18) then

$$ {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} = {}^{(\mathrm{x})}\dot{a}_{pq}\mathbf{g}^p\mathbf{g}^q = \overset{(\mathrm{x})}{\vdots}\left.\frac{\partial \boldsymbol{A}}{\partial t}\right|_{(X)} = \left.\frac{\partial a_{pq}}{\partial t}\right|_{(x)}\mathbf{g}^p\mathbf{g}^q + \left.\frac{\partial \boldsymbol{A}}{\partial x^s}\right|_{(x)}\left.\frac{\partial^{(\mathrm{B})}x^s}{\partial t}\right|_{(X)}, \tag{4.1}$$

$$\text{i.e.,}\quad {}^{(\mathrm{x})}\dot{a}_{pq} = \left.\frac{\partial a_{pq}}{\partial t}\right|_{(x)} + a_{pq;s}\,{}^{(\mathrm{x})}v^x, \tag{4.2}$$

- if $\left\{\widehat{x}^k\right\}$ is the defining coordinate system with respect to which the continuum moves according to (3.1) and with the with velocity ${}^{(\widehat{\mathrm{x}})}\mathbf{v}$ given by equation (3.22) then

$$ {}^{(\widehat{\mathrm{x}})}\boldsymbol{B}^{\cdot} = {}^{(\widehat{\mathrm{x}})}\dot{\widehat{b}}_{kl}\widehat{\mathbf{g}}^k\widehat{\mathbf{g}}^l = \overset{(\widehat{\mathrm{x}})}{\vdots}\left.\frac{\partial \boldsymbol{A}}{\partial t}\right|_{(X)} = \left.\frac{\partial \widehat{b}_{kl}}{\partial t}\right|_{(\widehat{x})}\widehat{\mathbf{g}}^k\widehat{\mathbf{g}}^l + \left.\frac{\partial \boldsymbol{B}}{\partial \widehat{x}^m}\right|_{(\widehat{x})}\left.\frac{\partial^{(\mathrm{B})}\widehat{x}^m}{\partial t}\right|_{(X)}, \tag{4.3}$$

$$\text{i.e.,}\quad {}^{(\widehat{\mathrm{x}})}\dot{\widehat{b}}_{kl} = \left.\frac{\partial \widehat{b}_{kl}}{\partial t}\right|_{(\widehat{x})} + \widehat{b}_{kl;m}\,{}^{(\widehat{\mathrm{x}})}\widehat{v}^m, \tag{4.4}$$

- if $\left\{X^K\right\}$ is the defining coordinate system in which the continuum does not move, i.e., the velocity ${}^{(\mathrm{X})}\mathbf{v} = \mathbf{0}$ then

$$ {}^{(\mathrm{X})}\boldsymbol{C}^{\cdot} = {}^{(\mathrm{X})}\dot{\widehat{c}}_{KL}\widehat{\mathbf{G}}^K\widehat{\mathbf{G}}^L = \overset{(\mathrm{X})}{\vdots}\left.\frac{\partial \boldsymbol{C}}{\partial t}\right|_{(X)} = \left.\frac{\partial \widehat{c}_{KL}}{\partial t}\right|_{(X)}\widehat{\mathbf{G}}^K\widehat{\mathbf{G}}^L, \tag{4.5}$$

$$\text{i.e.,}\quad {}^{(\mathrm{X})}\dot{\widehat{c}}_{KL} = \left.\frac{\partial \widehat{c}_{KL}}{\partial t}\right|_{(X)}. \tag{4.6}$$

Making use of the previous results, material time derivatives can be established for second-order tensors with position of indices other than above and for any tensor of higher order.

The material time derivatives obey the derivation rules valid for the sum and product of tensors.

Being real tensors the material time derivatives follow the general transformation laws of tensors. According to (2.5) for example:

$$^{(\mathrm{x})}\dot{\widehat{a}}^{k}{}_{l} = \frac{\partial \widehat{x}^k}{\partial^{(\mathrm{G})} x^p} \frac{\partial^{(\mathrm{G})} x^q}{\partial \widehat{x}^l} \, ^{(x)}\dot{a}^{p}{}_{q}. \tag{4.7}$$

4.2 The material time derivatives of the independent tensors $\boldsymbol{A}$**,** $\boldsymbol{B}$ and $\boldsymbol{C}$, which we have defined in a given coordinate system and discussed so far, are also independent of each other.

We are, however, faced with a distinct case when we consider the material time derivatives of the same tensor in various coordinate systems which move with respect to each other, i.e., if the tensor in question is defined independently of a coordinate system since, on the contrary, the material time derivative itself is always defined in a given coordinate system, as is the case, for instance, in respect of the material time derivatives of the tensor

$$\boldsymbol{A} = a_{pq}\mathbf{g}^p\mathbf{g}^q = \widehat{a}_{kl}\widehat{\mathbf{g}}^k\widehat{\mathbf{g}}^l = \widehat{a}_{KL}\widehat{\mathbf{G}}^K\widehat{\mathbf{G}}^L. \tag{4.8}$$

Depending on what the coordinate system is, the material time derivative of a tensor will be referred to as

- *material time derivative* if it is defined in the coordinate system $\{x^p\}$,
- *relative material time derivative* if it is defined in the coordinate system $\{\widehat{x}^k\}$,
- *convected material time derivative* if it is defined in the coordinate system $\{X^K\}$.

In addition, relations can be established between the various material time derivatives.

Assuming that formulae (4.2), (4.4) and (4.6) are valid for the tensor field given by (4.8), we obtain for the tensor $\boldsymbol{A}$, the material time derivative, the relative material time derivative and the convected material time derivative:

$$^{(\mathrm{x})}\dot{a}_{pq} = \left.\frac{\partial a_{pq}}{\partial t}\right|_{(x)} + a_{pq;s} \, ^{(\mathrm{x})}v^x, \tag{4.9}$$

$$^{(\widehat{\mathrm{x}})}\left(\widehat{a}_{kl}\right)^{\cdot} = \left.\frac{\partial \widehat{a}_{kl}}{\partial t}\right|_{(\widehat{x})} + \widehat{a}_{kl;m} \, ^{(\widehat{\mathrm{x}})}\widehat{v}^m, \tag{4.10}$$

$$^{(\mathrm{X})}\left(\widehat{a}_{KL}\right)^{\cdot} = \left.\frac{\partial \widehat{a}_{KL}}{\partial t}\right|_{(X)}. \tag{4.11}$$

The preceding equations could be used with minor changes concerning the position of indices for second-order tensors with indices positioned differently and for a tensor of any order.

4.3. Material time derivatives, defined respectively in the coordinate systems $\{x^p\}$ and $\{\widehat{x}^k\}$, $\{x^p\}$ and $\{X^K\}$, and finally in the coordinate systems $\{\widehat{x}^k\}$ and $\{X^K\}$

can be related to each other by means of the motion of grid and continuum in the coordinate system $\{x^p\}$, provided that the coordinate system $\{x^p\}$ is fixed.

Indeed, the material time derivative of the tensor (4.8) defined independently of the choice of a coordinate system can also be determined in the following manner:

$$
\begin{aligned}
{}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} &= {}^{(\mathrm{x})}\big(\boldsymbol{A}\left(\widehat{x}^1,\widehat{x}^2,\widehat{x}^3;t\right)\big)^{\cdot} = \overset{(\mathrm{x})}{\vdots} \frac{\partial\left(\widehat{a}_{kl}\widehat{\mathbf{g}}^k\widehat{\mathbf{g}}^l\right)}{\partial t}\bigg|_{(X)} = \\
&= \overset{(\mathrm{x})}{\vdots} \frac{\partial\left(\widehat{a}_{kl}\widehat{\mathbf{g}}^k\widehat{\mathbf{g}}^l\right)}{\partial t}\bigg|_{(\widehat{x})} + \frac{\partial \boldsymbol{A}}{\partial \widehat{x}^m}\frac{\partial^{(\mathrm{B})}\widehat{x}^m}{\partial t}\bigg|_{(X)} = \\
&= \frac{\partial \widehat{a}_{kl}}{\partial t}\bigg|_{(\widehat{x})}\widehat{\mathbf{g}}^k\widehat{\mathbf{g}}^l + \widehat{a}_{kl}\left(\overset{(\mathrm{x})}{\vdots}\frac{\partial\widehat{\mathbf{g}}^k}{\partial t}\bigg|_{(\widehat{x})}\widehat{\mathbf{g}}^l + \widehat{\mathbf{g}}^k \overset{(\mathrm{xx})}{\vdots} \frac{\partial\widehat{\mathbf{g}}^l}{\partial t}\bigg|_{(\widehat{x})}\right) + \frac{\partial \boldsymbol{A}}{\partial \widehat{x}^m}\frac{\partial^{(\mathrm{B})}\widehat{x}^m}{\partial t}\bigg|_{(X)}.
\end{aligned}
\tag{4.12}
$$

Substituting equations (4.10) and (2.19) we obtain

$$
{}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} = {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}_{**}\right)^{\cdot} - {}^{(\mathrm{Gx})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{Gx})}\boldsymbol{L}
$$

where ${}^{(\mathrm{Gx})}\boldsymbol{L}$ is the velocity gradient for the motion of grid. The asterisks which indicate the positions of indices, refer to the fact that the relation between the two material time derivatives depends on the positions of indices in the grid coordinate system $\left\{\widehat{x}^k\right\}$.

We may notice that the difference between ${}^{(\mathrm{x})}\boldsymbol{A}^{\cdot}$ and ${}^{(\widehat{\mathrm{x}})}\boldsymbol{A}^{\cdot}_{**}$ follows from the change of the base vectors $\widehat{\mathbf{g}}^k$ and $\widehat{\mathbf{g}}^l$ of the grid coordinate system $\left\{\widehat{x}^k\right\}$ in the fixed coordinate system $\{x^p\}$.

By repeating the above procedure for other positions of indices and gathering then the results we may write

$$
\begin{aligned}
&\text{I.} && {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}_{**}\right)^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{Gx})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{Gx})}\boldsymbol{L}, && (4.13)\\
&\text{II.} && {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}^{*}{}_{*}\right)^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{Gx})}\boldsymbol{L}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{Gx})}\boldsymbol{L}, && (4.14)\\
&\text{III.} && {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}_{*}{}^{*}\right)^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{Gx})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{Gx})}\boldsymbol{L}^{\mathrm{T}}, && (4.15)\\
&\text{IV.} && {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}^{**}\right)^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{Gx})}\boldsymbol{L}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{Gx})}\boldsymbol{L}^{\mathrm{T}}. && (4.16)
\end{aligned}
$$

In the case ${}^{(\mathrm{Gx})}\boldsymbol{D} = \boldsymbol{0}$, i.e., if the grid has a rigid body motion, ${}^{(\mathrm{Gx})}\boldsymbol{L} = {}^{(\mathrm{Gx})}\boldsymbol{W}$, where ${}^{(\mathrm{Gx})}\boldsymbol{W}$ is the spin tensor of the grid, equations (4.13)-(4.16) lead to the equations

$$
{}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}_{**}\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}^{*}{}_{*}\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}_{*}{}^{*}\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}^{**}\right)^{\cdot} = {}^{(\widehat{x})}\boldsymbol{A}^{\cdot}, \tag{4.17}
$$

$$
{}^{(\widehat{\mathrm{x}})}\boldsymbol{A}^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{Gx})}\boldsymbol{W}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{Gx})}\boldsymbol{W}. \tag{4.18}
$$

It is clear that the latter equation, which relates material time derivatives defined in the coordinate systems $\left\{\widehat{x}^k\right\}$ and $\{x^p\}$, is independent of the position of indices.

4.4. Apply all that has been said above to the metric tensors. In the case when $\{x^p\}$ is the fixed coordinate system then, according to (2.4), g_{pq} does not depend on time but $\widehat{g}_{kl}$ does, and conversely, when $\{\widehat{x}^k\}$ is the fixed coordinate system, then $\widehat{g}_{kl}$ does not depend on time but g_{pq} does. *Accordingly if, for instance,* $\{x^p\}$ *is the fixed coordinate system, then on the basis of equation (*4.13*) it follows*

$$
\begin{aligned}
{}^{(\mathrm{x})}\dot{g}_{pq}\frac{\partial^{(\mathrm{G})}x^p}{\partial\widehat{x}^k}\frac{\partial^{(\mathrm{G})}x^q}{\partial\widehat{x}^l} = 0 = {}^{(\widehat{\mathrm{x}})}\left(\widehat{g}_{kl}\right)^{\cdot} - {}^{(\mathrm{Gx})}\widehat{l}^s{}_k\widehat{g}_{sl} - \widehat{g}_{ks}\,{}^{(\mathrm{Gx})}\widehat{l}^s{}_l, \\
\text{i.e.}\quad {}^{(\widehat{\mathrm{x}})}\left(\widehat{g}_{kl}\right)^{\cdot} = {}^{(\mathrm{Gx})}\widehat{l}_{lk} + {}^{(\mathrm{Gx})}\widehat{l}_{kl} = 2\,{}^{(\mathrm{Gx})}\widehat{d}_{kl}\,. \qquad (4.19)
\end{aligned}
$$

Similarly, on the basis of equation (4.16) we have

$$
{}^{(\widehat{\mathrm{x}})}\left(\widehat{g}^{kl}\right)^{\cdot} = -\,2\,{}^{(\mathrm{Gx})}\widehat{d}^{kl}.
$$

4.5. Consider now the case of two grids moving arbitrarily with respect to each other. Let $\{\widehat{x}^k\}$ and $\{\widehat{\xi}^b\}$ be the two grids. Further let ${}^{(\xi\widehat{\mathrm{x}})}\boldsymbol{L} = {}^{(\xi\mathrm{x})}\boldsymbol{L} - {}^{(\mathrm{Gx})}\boldsymbol{L}$, where ${}^{(\xi\mathrm{x})}\boldsymbol{L}$ and ${}^{(\mathrm{Gx})}\boldsymbol{L}$ are the velocity gradients in the coordinate systems $\{\widehat{\xi}^b\}$ and $\{\widehat{x}^k\}$ being measured in the coordinate system $\{x^p\}$. In other words ${}^{(\xi\widehat{\mathrm{x}})}\boldsymbol{L}$ is the gradient of the velocity ${}^{(\xi\widehat{\mathrm{x}})}\mathbf{v} = {}^{(\xi\mathrm{x})}\mathbf{v} - {}^{(\mathrm{Gx})}\mathbf{v}$, which we measure observing the motion of the coordinate system $\{\widehat{\xi}^b\}$ from the coordinate system $\{\widehat{x}^k\}$. Writing the equations (4.13)-(4.16) both for the coordinate system $\{\widehat{x}^k\}$ and for the coordinate system $\{\widehat{\xi}^b\}$ and then subtracting the equations resulting from each other we obtain:

$$
\begin{aligned}
&\text{I.} \quad {}^{(\widehat{\xi})}\left(\boldsymbol{A}_{**}\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}_{**}\right)^{\cdot} + {}^{(\xi\widehat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\xi\widehat{\mathrm{x}})}\boldsymbol{L}, && (4.20)\\
&\text{II.} \quad {}^{(\widehat{\xi})}\left(\boldsymbol{A}^*{}_*\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}^*{}_*\right)^{\cdot} - {}^{(\xi\widehat{\mathrm{x}})}\boldsymbol{L}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\xi\widehat{\mathrm{x}})}\boldsymbol{L}, && (4.21)\\
&\text{III.} \quad {}^{(\widehat{\xi})}\left(\boldsymbol{A}_*{}^*\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}_*{}^*\right)^{\cdot} + {}^{(\xi\widehat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\xi\widehat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}, && (4.22)\\
&\text{IV.} \quad {}^{(\widehat{\xi})}\left(\boldsymbol{A}^{**}\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\left(\boldsymbol{A}^{**}\right)^{\cdot} - {}^{(\xi\widehat{\mathrm{x}})}\boldsymbol{L}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\xi\widehat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}. && (4.23)
\end{aligned}
$$

The results implied in equations (4.13)-(4.16) and (4.20)-(4.23) can be summarized as follows:

If the material time derivative of a tensor defined in a given - say, in the first - coordinate system is known, then the material time derivative of the tensor defined in another - say, the second - coordinate system (moving arbitrarily with respect to the first one) is obtained by adding such an expression to the first material time derivative which is a linear combination of products involving as factors the gradient of the velocity vector field – measured observing the motion of the second coordinate system relative to the first one – and the tensor itself. The terms involved in the linear combination depend on the order of the tensor and the positions of indices.

4.6. On the basis of the above rule it holds for time rate of change of tensors defined in the convected coordinate system (without detailing the equations with mixed positions of indices) that

$$\text{I.}\quad {}^{(\mathrm{X})}\left(\boldsymbol{A}_{**}\right)^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} + {}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{L}, \tag{4.24}$$

$$\ldots$$

$$\text{IV.}\quad {}^{(\mathrm{X})}\left(\boldsymbol{A}^{**}\right)^{\cdot} = {}^{(\mathrm{x})}\boldsymbol{A}^{\cdot} - {}^{(\mathrm{x})}\boldsymbol{L}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\mathrm{x})}\boldsymbol{L}^{\mathrm{T}}. \tag{4.25}$$

where ${}^{(\mathrm{x})}\boldsymbol{L}$ is the velocity gradient for the velocity vector field ${}^{(\mathrm{x})}\mathbf{v}$, and

$$\text{I.}\quad {}^{(\mathrm{X})}\left(\boldsymbol{A}_{**}\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\boldsymbol{A}^{\cdot} + {}^{(\widehat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot{}^{(\widehat{\mathrm{x}})}\boldsymbol{L}, \tag{4.26}$$

$$\ldots$$

$$\text{IV.}\quad {}^{(\mathrm{X})}\left(\boldsymbol{A}^{**}\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\boldsymbol{A}^{\cdot} - {}^{(\widehat{\mathrm{x}})}\boldsymbol{L}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot{}^{(\widehat{\mathrm{x}})}\boldsymbol{L}^{\mathrm{T}}. \tag{4.27}$$

where ${}^{(\widehat{\mathrm{x}})}\boldsymbol{L}$ is the velocity gradient for the velocity vector field ${}^{(\widehat{\mathrm{x}})}\mathbf{v}$.

4.7. Relations between the time derivatives can be given, of course, in indicial notations. Considering relation (4.14), for example, we may write

$$^{(\widehat{\mathrm{x}})}\left(\widehat{a}^{k}{}_{l}\right)^{\cdot} = \left({}^{(\mathrm{x})}\dot{a}^{p}{}_{q} - {}^{(\mathrm{Gx})}l^{p}{}_{s}a^{s}{}_{q} + a^{p}{}_{s}\,{}^{(\mathrm{Gx})}l^{s}{}_{q}\right)\frac{\partial\widehat{x}^{k}}{\partial^{(\mathrm{G})}x^{p}}\frac{\partial^{(\mathrm{G})}x^{q}}{\partial\widehat{x}^{l}}\,. \tag{4.28}$$

The results obtained can also be generalized for a tensor of any order. Considering a third-order tensor $\boldsymbol{A} = a_{p}{}^{qr}\mathbf{g}^{p}\mathbf{g}_{q}\mathbf{g}_{r} = \widehat{a}_{k}{}^{lm}\widehat{\mathbf{g}}^{k}\widehat{\mathbf{g}}_{l}\widehat{\mathbf{g}}_{m}$, for example, we shall find

$$^{(\widehat{\mathrm{x}})}\left(\widehat{a}_{k}{}^{lm}\right)^{\cdot} = \left({}^{(\mathrm{x})}\dot{a}_{p}{}^{qr} + {}^{(\mathrm{Gx})}l^{s}{}_{p}a_{s}{}^{qr} - a_{p}{}^{sr}\,{}^{(\mathrm{Gx})}l^{q}{}_{s} - a_{p}{}^{qs}\,{}^{(\mathrm{Gx})}l^{r}{}_{s}\right)\frac{\partial^{(\mathrm{G})}x^{p}}{\partial\widehat{x}^{k}}\frac{\partial\widehat{x}^{l}}{\partial^{(\mathrm{G})}x^{q}}\frac{\partial\widehat{x}^{m}}{\partial^{(\mathrm{G})}x^{r}}\,. \tag{4.29}$$

4.8. Dependence of material time derivatives on position of indices can also be shown in indicial notations. For this purpose we write equation (4.28) in the form

$$^{(\mathrm{x})}\dot{a}^{r}{}_{q} = \frac{\partial^{(\mathrm{G})}x^{r}}{\partial\widehat{x}^{m}}\frac{\partial\widehat{x}^{l}}{\partial^{(\mathrm{G})}x^{q}}\,{}^{(\widehat{\mathrm{x}})}\left(\widehat{a}^{m}{}_{l}\right)^{\cdot} + {}^{(\mathrm{Gx})}l^{r}{}_{s}a^{s}{}_{q} - a^{r}{}_{s}\,{}^{(\mathrm{Gx})}l^{s}{}_{q}\,.$$

Multiplying both sides by g_{pr} and manipulating then the first term on the right side into

$$g_{pr}\frac{\partial^{(\mathrm{G})}x^{r}}{\partial\widehat{x}^{m}}\frac{\partial\widehat{x}^{l}}{\partial^{(\mathrm{G})}x^{q}}\,{}^{(\widehat{\mathrm{x}})}\left(\widehat{a}^{m}{}_{l}\right)^{\cdot} = \widehat{g}_{km}\frac{\partial\widehat{x}^{k}}{\partial^{(\mathrm{G})}x^{p}}\frac{\partial\widehat{x}^{l}}{\partial^{(\mathrm{G})}x^{q}}\,{}^{(\widehat{\mathrm{x}})}\left(\widehat{a}^{m}{}_{l}\right)^{\cdot} =$$
$$= \frac{\partial\widehat{x}^{k}}{\partial^{(\mathrm{G})}x^{p}}\frac{\partial\widehat{x}^{l}}{\partial^{(\mathrm{G})}x^{q}}\left({}^{(\widehat{\mathrm{x}})}\left(\widehat{g}_{km}\widehat{a}^{m}{}_{l}\right)^{\cdot} - \left(\widehat{a}_{km}\right)^{\cdot}\widehat{a}^{m}{}_{l}\right)\,,$$

we obtain, also with regard to equation (4.19) that

$$^{(\mathrm{x})}\dot{a}_{pq} = \frac{\partial\widehat{x}^{k}}{\partial^{(\mathrm{G})}x^{p}}\frac{\partial\widehat{x}^{l}}{\partial^{(\mathrm{G})}x^{q}}\,{}^{(\widehat{\mathrm{x}})}\left(\widehat{a}_{kl}\right)^{\cdot} - {}^{(\mathrm{Gx})}l^{s}{}_{p}a_{sq} - a_{ps}\,{}^{(\mathrm{Gx})}l^{s}{}_{q}$$

which is identical to equation (4.13).

4.9. Now we shall consider a covariant and a contravariant vector:

$$^{(\widehat{\mathrm{x}})}\dot{\mathbf{a}}_{*} = {}^{(\mathrm{x})}\dot{\mathbf{a}} + \mathbf{a}\cdot{}^{(\mathrm{Gx})}\boldsymbol{L}, \qquad {}^{(\widehat{\mathrm{x}})}\dot{\mathbf{a}}^{*} = {}^{(x)}\dot{\mathbf{a}} - {}^{(\mathrm{Gx})}\boldsymbol{L}\cdot\mathbf{a}. \tag{4.30}$$

4.10. Let ${}^{(\mathrm{x})}\mathbf{a}$ and ${}^{(\widehat{\mathrm{x}})}\mathbf{a}^*$ be the accelerations of the point X^K in the coordinate systems $\{x^p\}$ and $\{\widehat{x}^k\}$. For completeness we shall give how these accelerations are related to each other.

By definition

$$ {}^{(\mathrm{x})}\mathbf{a} = {}^{(\mathrm{x})}\left({}^{(\mathrm{x})}\mathbf{v}\right)^{\cdot} \quad \text{and} \quad {}^{(\widehat{\mathrm{x}})}\mathbf{a} = {}^{(\widehat{\mathrm{x}})}\left({}^{(\widehat{\mathrm{x}})}\mathbf{v}\right)^{\cdot}. \tag{4.31} $$

It follows from equation (3.26) that

$$ {}^{(\mathrm{x})}\mathbf{a} = {}^{(\mathrm{x})}\left({}^{(\mathrm{x})}\mathbf{v}\right)^{\cdot} = {}^{(\mathrm{x})}\left({}^{(\mathrm{Gx})}\mathbf{v} + {}^{(\widehat{\mathrm{x}})}\mathbf{v}\right)^{\cdot}, $$

where

$$ {}^{(\mathrm{x})}\left({}^{(\mathrm{Gx})}\mathbf{v}\right)^{\cdot} = \overset{(\mathrm{x})}{:} \left.\frac{\partial^{(\mathrm{Gx})}\mathbf{v}}{\partial t}\right|_{(X)} = \overset{(\mathrm{x})}{:} \left.\frac{\partial^{(\mathrm{Gx})}\mathbf{v}}{\partial t}\right|_{(\widehat{x})} + \left.\frac{\partial^{(\mathrm{Gx})}\mathbf{v}}{\partial \widehat{x}^k}\,\frac{\partial^{(\mathrm{B})}\widehat{x}^k}{\partial t}\right|_{(X)} = $$

$$ = {}^{(\mathrm{Gx})}\mathbf{a} + \left({}^{(\mathrm{Gx})}\mathbf{v}\nabla\right)\cdot\ {}^{(\widehat{\mathrm{x}})}\mathbf{v}\,. $$

According to equation (4.30) we have

$$ {}^{(\mathrm{x})}\left({}^{(\widehat{\mathrm{x}})}\mathbf{v}\right)^{\cdot} = {}^{(\widehat{\mathrm{x}})}\left({}^{(\widehat{\mathrm{x}})}\mathbf{v}\right)^{\cdot} + {}^{(\mathrm{Gx})}\boldsymbol{L}\cdot{}^{(\widehat{\mathrm{x}})}\mathbf{v}. $$

On the basis of the above equations we get from equation (4.31)

$$ {}^{(\mathrm{x})}\mathbf{a} = {}^{(\widehat{\mathrm{x}})}\mathbf{a}^* + {}^{(\mathrm{Gx})}\mathbf{a} + \mathbf{2}{}^{(\mathrm{Gx})}\boldsymbol{L}\cdot\ {}^{(\widehat{\mathrm{x}})}\mathbf{v}. \tag{4.32} $$

For our latter considerations we remark that neither the velocity ${}^{(\mathrm{x})}\mathbf{v}$ nor the acceleration ${}^{(\mathrm{x})}\mathbf{a}$ are physically objectiv quantities.

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