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Thermoelastic Problems of Functionally Graded and Composite Structural Components

Synopsis of PhD theses by
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Main Topic Group: Fundamental Sciences in Mechanical Engineering
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1. Preliminaries

In recent decades, composite and functionally graded materials have been widely used in numerous engineering applications due to their excellent material properties. Composites are created by physically combining different materials in solid state which usually results in a light-weighted material with great mechanical properties and sharp interfaces between the constituent materials. The utilization of composites is limited because under extreme working conditions a phenomenon called delamination occurs. This process is especially problematic in high temperature environments when the parent materials have different coefficients of linear thermal expansion. Functionally graded materials (*FGM*s) are advanced materials in which the composition and structure gradually change resulting in a corresponding change in the properties of the material. In functionally graded materials the sharp interfaces between the constituent materials are eliminated -which is where failure can be initiated- and replaced with a gradient interface which produces smooth transition from one material to the next.

A lot of books and papers deal with the thermoelastic problems of homogeneous, isotropic materials. In the past few years many researchers investigated the mechanics of structures made from inhomogeneous materials, such as composite and functionally graded materials. The concept and the mechanics of functionally graded materials have become more popular in Europe.

Lots of works deal with the mechanics of functionally graded materials from various aspects, for example finite element modelling, or the stress, stability, dynamic analysis and fracture mechanics mostly for FGM beams, plates and shells [1, 2]. The dissertation deals with the steady-state thermoelastic problems of simple composite and functionally graded structural components, such as disks, spheres and beams, which are subjected to steady-state thermal and mechanical loadings. As regards the mechanics of these problems, investigations began in the 1990s.

At first, works like [3,4] considered specific loadings, such as only thermal or only mechanical loadings. Papers [4-7] investigated simple radially graded components where the material properties were described by simple power-law functions. Nayak et al. [6] elaborated an analytical solution to obtain the radial, tangential and effective stresses within thick spherical pressure vessels made of FGMs subjected to axisymmetric mechanical and thermal loadings. The properties of the material for the vessel are assumed to be graded in the radial direction based on a power-law function of the radial coordinate but Poisson's ratio has constant value. With thermal boundary conditions of the third kind and steady-state unidirectional radial heat conduction, the equilibrium equation reduces to the Navier equation. A work by Bayat, Mahdi and Torabi [7] dealt with the previously presented problem too, and investigated the effect of the index parameter of the power-law functions on the stress distribution. Papers [8-11] dealt with the thermoelastic problems of functionally graded tubes and spherical pressure vessels with exponential material distribution.

Several works [12-17] give detailed analysis of the thermal stress problems for homogeneous isotropic elastic disks with axisymmetric temperature field. Furthermore these books and papers neglect the convective heat transfer on the lower and upper plane surfaces of the disks. Numerous papers, such as [18-22], presented thermomechanical problems of functionally graded disks but the material parameters and the geometry were special functions of the radial coordinate. In paper by Pen and Li [23] the thermoelastic problem of isotropic functionally graded disks with arbitrary radial inhomogeneity was considered. The numerical solution of the steady-state thermoelastic problem is reduced to a solution of a Fredholm integral equation. Most of these works neglected the temperature dependency of functionally graded materials.

As for functionally graded beams, paper [24] dealt with the two-dimensional problem of exponentially graded beams under uniaxial tension and bending using the Airy stress function method together with the strain compatibility equation. Sankar [25] studied a bending problem of a simply supported FGM beam based on the theories of beams and two-dimensional elasticity. Zhong and Yu [26] obtained the general solution for a cantilever made from functionally graded beam subjected to different kinds of loads. The paper by Ying et al. [27] gave a two-dimensional elasticity solution for functionally graded beams resting on elastic foundations. Wang and Liu [28] analysed a bimaterial beam with graded intermediate layer subjected to uniform loading on the upper surface. A paper by Li et al. [29] made a stress analysis of FGM beams using effective principal axes. Papers [30-33] dealt with the problem of bimetallic strips made from two different homogeneous components, although the curved beams were not investigated. The thermoelastic pure bending problem of nonhomogeneous prismatic bars, based on the Euler-Bernoulli beam theory was analysed by Stokes [34]. Stokes's paper used a direct approach starting from the assumed form of normal strain.

2. Objectives

As we have seen, analytical solutions were derived only in special cases, for example for power-law based material properties, constant Poisson's ratio etc. Most of the papers and works neglect the temperature dependency of the material properties. The aim of the dissertation is to deal with the problems of functionally graded simple structural components made from isotropic functionally graded materials and layered -laminated-composites subjected to thermal and mechanical loads. I intended to derive methods for thermoelastic problems of functionally graded materials with properties described by arbitrary spatial and temperature- dependent functions. I investigated stationary or steady-state thermoelastic problems. The time-independence of the functions involved separates the analysis of the temperature field from that of the elastic field, therefore these problems become uncoupled. In many cases the functionally graded structural components are built with additive methods layer by layer, therefore we can approximate the problem of functionally graded components with methods of layered components with finite layer number n. Obviously when $n \to \infty$ we get to the functionally graded materials. In view of this, the objectives of the dissertation are

- to derive analytical methods to calculate the temperature field, displacements and stress field within layered spherical bodies;
- to develop fast and accurate numerical methods for determining the temperature-, displacement- and stress field within functionally -radially- graded spherical components, when the material properties are arbitrary functions of the radial coordinate and temperature;
- investigation of special problems of functionally graded spherical bodies, such as incompressible or piezoelectric, radially polarized materials;
- to present analytical solutions for functionally graded spheres, then compare the developed one-dimensional numerical and analytical methods to each other and finite element simulations;
- to develop analytical solutions for calculating the temperature field, displacements and normal stresses in layered composite disks;
- derivation of numerical methods for rotating thin functionally graded disks with arbitrary thickness profile when the material properties are arbitrary functions of the radial coordinate and the temperature field, furthermore there are combined thermal and mechanical loads on the cylindrical boundary surfaces;
- to determine the thermal stresses and displacements in non-homogeneous prismatic bars caused by mechanical and thermal loads when the cross section of the bar is an arbitrary bounded plain domain, the material properties and the temperature field do not depend on the axial coordinate;
- to deal with the problem of curved layered composite and functionally graded beams subjected to special thermal and mechanical loads;
- to compare the developed methods to each other, to finite element solutions and to results of the open literature.

3. Investigations performed

I dealt with thermoelastic problems of layered composite and functionally graded simple components, such as spheres, disks, layered tubes and beams subjected to thermal and mechanical loads. The time-independence of the functions divides the problems into a heat conduction- and an elasticity problem.

As for the heat conduction problems, my aim was to determine the axisymmetric temperature field in spherical bodies and disks. It was assumed that the heatflow and the temperature are all continuous functions of the radial coordinate. Analytical solutions were presented for layered spherical bodies with axisymmetric thermal boundary conditions of first and third kinds and for radially graded disk and spheres with specific material distribution. Based on this multilayered approach, the application of the Kirchoff integral transformation and discretization considerations, a numerical solution was presented to calculate the temperature field in radially graded spheres when the material parameters are arbitrary functions of the radial coordinate and temperature. In the case of thin disks, three models were derived to determine the axisymmetric temperature field. In the first heat conduction problem a multilayered approach was investigated with thermal boundary conditions of the first kind on the cylindrical boundary surfaces and arbitrary radial coordinate-dependent convective heat transfer on the lower and upper plane boundary surfaces. For disks with constant thickness and homogeneous layers, these equations give the analytical solution. In my second model the method of finite differences was used to solve the heat conduction problem of functionally graded disks with arbitrary thickness profile and boundary conditions of the third kind, where the parameters depend on the radial coordinate and on the temperature. The last method presented a solution for functionally graded disks with variable thickness and temperature-dependent material parameters which is similar to the one derived for the spherical bodies. I have investigated the possibilities of the approximation of the temperature field. The results of these models were used for the further calculations.

I have derived two analytical methods to determine the displacements and stress field in layered composite spherical bodies consisting of perfectly bonded homogeneous layers subjected to axisymmetric mechanical and thermal loads exerted on the inner and outer boundary surfaces. The first model used a direct form of the displacement field, which is the exact solution of the Navier equation and fits the displacement values at the boundary surfaces of the layers. The second method derived the solutions from the superposition of the cases when there is only mechanical load and when there is only thermal load, the validity of the solution comes from the linearity of the field equations and boundary conditions. Based on these equations I have investigated the possibilities of modelling radially graded spheres with the methods developed for layered composite spheres. For radially graded spheres with arbitrary radial coordinate and temperature-dependent material properties, a numerical model was elaborated which uses a coupled system of ordinary differential equations containing the radial displacement and stress function and transforms the two point boundary value problem to an initial value problem. Furthermore, I have derived a similar numerical solution for radially graded piezoelectric spherical actuators, where the solution comes from the combination of the numerical solutions of 5 initial value problems.

An analytical solution was presented for the case when the distribution of the Young modulus is described by a power-law function, the coefficient of thermal expansion depends on the radial coordinate and on the temperature in a prescribed way. I solved the thermoelastic problem of radially graded spheres with stress function when the material properties follow a power-law distribution, moreover an additional method was elaborated for incompressible functionally graded spheres.

I have derived two numerical methods for the thermoelastic analysis of thin functionally graded rotating disks -with arbitrary thickness profile- subjected to combined axisymmetric thermal and mechanical loads. The temperature-dependent material properties of rotating disks vary arbitrarily along the radial coordinate. In the first method the displacements and the normal stresses are determined by a multilayered approach which can be used as an analytical solution for layered composite disks with constant thickness. This model solves the original problem as the superposition of two separate problems - when the disk is subjected to thermal load only and when there is only constant mechanical load exerted to the cylindrical boundary surfaces. This method was modified to tackle some thermoelastic problems of multilayered tubes loaded with constant temperature field and pressure where the equations of generalized plane strain were utilized. The tube consists of radially bonded homogeneous layers without axial bonding. The second developed method used a coupled system of ordinary differential equations containing the radial displacement and stress function which transforms the two point boundary value problem to an initial value problem. The solution of this system of linear differential equations comes from the combination of three initial value problems calculated with Runge-Kutta method. An analytical method -using stress functions- was proposed for the case when the distribution of the material properties are prescribed as power-law functions of the radial coordinate.

As for beams, methods were derived to determine the displacement- and stress field of functionally graded prismatic bars whose cross section is an arbitrary bounded plain domain, the temperature field and the material properties are arbitrary functions of the cross-sectional coordinates but do not vary in the axial direction. I have derived a model based on the principle of minimum of complementary energy when the prismatic bar is subjected to certain mechanical and thermal loads. For the equations the E-weighted centre line was introduced, and the method of Lagrange multipliers was utilized. Furthermore a method was presented using a direct form of the axial normal strain -similarly to [34]- in the case of eccentric tensile load and the equations of the displacement field were determined based on the Euler-Bernoulli beam theory.

I have presented the equations for layered curved beams and strips and I have focused on the problem of bimetallic curved beam in uniform temperature field. One model based on the theory of generalized plane stress of elasticity and uses stress functions, the second method uses a strength of materials approach as in [35]. The method was extended to approximate the thermoelastic behaviour of functionally -radially- graded curved strips.

Several numerical examples were presented. The programing and the numerical calculations were executed in Maple 15. The developed methods were verified by comparing them to each other, to results obtained from the open literature and to finite element solutions –in Abaqus CAE. It turned out, that the presented methods are in good agreement with the literature and finite element models, furthermore they are fast and in many cases have high accuracy.

4. Novel results

Thesis 1

I have derived two analytical methods to determine the displacements and stress field in layered composite spherical bodies subjected to axisymmetric mechanical and thermal loads exerted on the inner and outer boundary surfaces. The homogeneous layers were perfectly bonded. The first method uses a direct form of the displacement field, the second model derives the solutions of the combined loading from the superposition of the cases when there is only mechanical load and when there is only thermal load. An analytical solution is presented for the case when the distribution of the Young modulus is described by a certain power-law function, moreover the coefficient of thermal expansion depends on the radial coordinate and on the temperature in a prescribed way. I have solved the thermoelastic problem of radially graded spheres with stress function when the material properties follow a power-law distribution. I have investigated the possibilities of modelling the functionally graded spheres with the method of layered composite spheres. The developed methods have been verified by data obtained from the literature and comparisons have been made with each other and they have led to the same results.

Thesis 2

I have elaborated two numerical methods to deal with the thermoelastic problem of functionally graded spherical bodies subjected to axisymmetric thermal loading and constant pressure. The temperature field, displacements and normal stresses are determined when the material properties are arbitrary functions of the radial coordinate and temperature. The first model is based on the multilayered approach of Thesis 1. The second method uses a coupled system of ordinary differential equations containing the radial displacement and stress function and transforms the two point boundary value problem to an initial value problem. I have derived a numerical solution for radially graded piezoelectric spherical actuators and an analytical method for incompressible functionally graded spheres. By means of numerical examples the accuracy of the developed numerical methods have been investigated, compared to the analytical solutions of Thesis 1 and have been verified by finite element simulations. According to these, it turns out that the numerical models have good accuracy.

Thesis 3

I have derived two numerical methods for the thermoelastic analysis of thin functionally graded rotating disks subjected to combined axisymmetric thermal and mechanical loads. The temperature-dependent material properties of the rotating disk vary arbitrarily along the radial coordinate, moreover the thickness of the disk is an arbitrary function of the radial coordinate. The equations of the steady-state temperature fields have been presented for three cases with different thermal boundary conditions. In the first novel method the displacements and the normal stresses are determined by a multilayered approach which can be used as an analytical solution for layered composite disks with constant thickness. This method has been modified to tackle some thermoelastic problems of multilayered tubes which consist of radially bonded homogeneous layers. Furthermore, the tubes are loaded with constant temperature field and pressure. The second developed method uses a coupled system of ordinary differential equations containing the radial displacement and stress function which transforms the two point boundary value problem to an initial value problem. An analytical method is proposed for the case when the distribution of the material properties are prescribed as power-law functions of the radial coordinate. The developed numerical methods have been compared to my analytical solution and finite element simulations. The results shows that the models have high accuracy.

Thesis 4

I have elaborated methods to determine the displacement- and stress field of functionally graded prismatic bars whose cross section is an arbitrary bounded plain domain. The material properties and the temperature field are arbitrary functions of the cross-sectional coordinates and do not vary in the axial direction. I have derived a model based on the principle of minimum of complementary energy for the case, when the prismatic bar is subjected to certain mechanical and thermal loads. Furthermore a method has been developed using a direct form of the axial normal strain. I have presented the equations for layered curved beam and I have focused on the problem of bimetallic curved beam in uniform temperature field. The method was extended to approximate the thermoelastic behaviour of functionally -radially- graded curved strips. The developed methods were verified by literature and finite element simulations.

5. Possible applications and future research

The results achieved can be applied to the design of simple functionally graded components. Especially the optimization of the material distribution within functionally graded members could become easier with the developed numerical models, it may reduce the development time and save costs. Moreover, these models can be used for benchmark purposes in order to verify the results of other numerical methods.

Some of the results could be harnessed in the education, as nowadays these modern materials are gradually gathering ground. For example the calculation of effective stresses in layered composite parts and the simple boundary value methods can be interesting for engineering students.

As for the future research, it would be interesting to verify the calculations with experiments. It is an interesting question what the connection is between the fabrication processes, the material distribution and the developed models. Developing finite element models for these problem might also be worthy to deal with.

Based on the presented methods, several additional improvements and generalizations could be made. It would be interesting to deal with non-axisymmetric problems of spherical pressure vessels and disk, or with time-dependent coupled thermoelastic problems.

6. The candidate's relevant publications

The following publications were made in the topic of the dissertation:

Articles in journals:

- Gönczi D., Ecsedi I.: Thermoelastic analysis of functionally graded hollow circular disk. Archieve of Mechanical Engineering, Vol 62. (1), pp. 5-18., 2015.
- Gönczi D., Ecsedi I.: Thermoelastic stresses in nonhomogeneous prismatic bars. Annals of Faculty of Engineering Hunedoara – International Journal of Engineering, Vol 13. (2), pp. 49-52., 2015.
- Gönczi D.: Thermoelastic analysis in functionally graded incompressible spherical bodies. Annals of Faculty of Engineering Hunedoara – International Journal of Engineering, Vol 14. (2), pp. 103-106., 2016.
- Gönczi D., Ecsedi I.: Hőokozta feszültségek és elmozdulások számítása gömb alakú kivágással gyengített végtelen kiterjedésű rugalmas testben (Determination of thermal stresses and displacements in infinite elastic media with spherical cutting-out, in Hungarian). Multidiszciplináris Tudományok: A Miskolci Egyetem Közleménye, Vol. 3.(2), pp. 279-288., 2013.
- Gönczi D., Ecsedi I.: Hőfeszültségek számítása üreges hengeres testekben hőmérséklettől függő anyagállandók esetén (Thermal stresses in hollow cylindrical bodies with temperature dependent material properties, in Hungarian). GÉP, Vol. 64.(5), pp. 28-32., 2013.
- Gönczi D., Ecsedi I.: Hőokozta feszültségek és elmozdulások meghatározása rétegzett körhenger alakú testekben (Determination of thermal stresses and displacements in layered cylindrical bodies, in Hungarian). Multidiszciplináris Tudományok: A Miskolci Egyetem Közleménye, Vol 2. (1), pp. 39-48., 2012.

Conference Papers:

- Gönczi D.: The determination of displacements stresses in functionally graded hollow spherical bodies. XXIX. MicroCAD International Scientific Conference, Miskolci Egyetem (ME), Paper D2, 2015.
- Gönczi D.: The determination of displacement field and normal stresses in multilayered spherical bodies. XXVIII. MicroCAD International Scientific Conference, Miskolci Egyetem (ME), Paper D35, 2014.
- Gönczi D.: Thermoelastic analysis of layered disks. XXXVIII. MicroCAD International Scientific Conference, Miskolci Egyetem (ME), Paper D36, 2014.
- Gönczi D., Ecsedi I.: Analysis of Bimetallic Circular Plate. Spring Wind Conference, Műszaki szekció, pp. 107-117., 2014.
- Gönczi D., Ecsedi I.: The Determination of Thermoelastic Stresses and DispIacement in Layered Spherical Bodies. XXVII. MicroCAD International Scientific Conference. Miskolci Egyetem (ME), Paper 6., 2013.
- Gönczi D., Ecsedi I.: Modelling of steady-state heat conduction and thermal stresses in a hollow cylinder. 8th International Conference of PhD Students, Paper F3, 2012.

Other works:

- Gönczi D.: Thermomechanical analysis of simple structural components subjected to mechanical and thermal loads. PhD students' Seminar, Report for István Sályi Doctoral School, pp. 1-27., 2014.
- Gönczi D.: Hő és mechanikai terhelésnek alávetett tárcsák, üreges gömb és körhenger alakú testek vizsgálata (Mechanical analysis of disks, spherical and cylindrical bodies subjected to mechanical and thermal loads, in Hungarian). PhD students' Seminar, Report for István Sályi Doctoral School, pp. 1-24., 2013.
- Gönczi D.: Speciális feladatok végeselemes modellezése I. (Finite element modelling of special problems I., in Hungarian) University of Miskolc, pp. 1-57., 2013.

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