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APPROXIMATE CALCULATION OF IMPROPER INTEGRALS

by

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For the evaluation of improper integrals several methods are available. A well known one is Gaussian quadrature [1]. However, when using this method one may encounter the difficulties of finding the suitable weight function, the polinomials orthogonal with respect to this function, and the zeros of these polinomials, as well. The object of this paper is to present a practical method for the evaluation of certain type improper integrals, using of which arbitrarily good approximation can be achieved.

Consider the integral

$$I = \int_{x_1}^{x_2} \frac{dx}{[f(x)]^b} \qquad (0 < b < 1), \qquad (1)$$

where the function $f(x)$ is Riemann-integrable on the interval (x_1, x_2) and positive on this interval, furthermore $f(x_1) = 0, f'(x_1) > 0$ and $f(x_2) > 0$. The following theorem will be proved for the integrals of above type.

THEOREM. *If there exists a number $\epsilon > 0$ such that $f''(x)$ exists and $f''(x) \neq 0$ on the interval $(x_1, x_1 + \epsilon)$, then the integral (1) can be approximated as follows*

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a) $$R_1(\epsilon)+\int_{x_1+\epsilon}^{x_2}\frac{dx}{[f(x)]^b}>I>R_2(\epsilon)+\int_{x_1+\epsilon}^{x_2}\frac{dx}{[f(x)]^b}\ ;$$

if $f''(x)>0, \quad x\in(x_1,x_1+\epsilon)$

b) $$R_1(\epsilon)+\int_{x_1+\epsilon}^{x_2}\frac{dx}{[f(x)]^b}<I<R_2(\epsilon)+\int_{x_1+\epsilon}^{x_2}\frac{dx}{[f(x)]^b}\ ;$$

if $f''(x)<0, \quad x\in(x_1,x_1+\epsilon)$,

where

$$R_1(\epsilon)=\frac{\epsilon^{1-b}}{(1-b)[f'(x_1)]^b} \tag{2}$$

and

$$R_2(\epsilon)=\frac{\epsilon}{(1-b)[f(x_1+\epsilon)]^b}\ . \tag{3}$$

PROOF. We introduce the functions

$$f_1(x)=\begin{cases} f'(x_1)(x-x_1)\ , & x_1\leqslant x\leqslant x_1+\epsilon \\ f(x) \ , & x_1+\epsilon<x\leqslant x_2 \end{cases} \tag{4}$$

and

$$f_2(x)=\begin{cases} \dfrac{f(x_1+\epsilon)}{\epsilon}(x-x_1)\ , & x_1\leqslant x\leqslant x_1+\epsilon \\ f(x) \ , & x_1+\epsilon<x\leqslant x_2 \end{cases} \tag{5}$$

It is easy to see that in these formulae $f(x)$ was replaced by its tangent and chord on the segment $[x_1, x_1+\epsilon]$. Using the formulae (1) – (5), we have

$$I_1=\int_{x_1}^{x_2}\frac{dx}{[f_1(x)]^b}=R_1(\epsilon)+\int_{x_1+\epsilon}^{x_2}\frac{dx}{[f(x)]^b} \tag{6}$$

and

$$I_2=\int_{x_1}^{x_2}\frac{dx}{[f_2(x)]^b}=R_2(\epsilon)+\int_{x_1+\epsilon}^{x_2}\frac{dx}{[f(x)]^b}\ . \tag{7}$$

Since one of the functions $f_1(x)$, $f_2(x)$ is a majorant and the other one is a minorant of $f(x)$ on the segment $[x_1, x_1+\epsilon]$, therefore one of the integrals (6), (7) can be considered as a lower bound, the other as an upper one of integral (1). It can be easily verified that

$$I_1 > I > I_2 \; ; \quad \text{if} \quad f''(x) > 0 , \quad x \in (x_1, x_1 + \epsilon)$$

and

$$I_1 < I < I_2 \; ; \quad \text{if} \quad f''(x) < 0 , \quad x \in (x_1, x_1 + \epsilon) \; .$$

(6) (7)

Subtracting integral (4) from (5), we obtain

$$I_2 - I_1 = R_2(\epsilon) - R_1(\epsilon) \, . \tag{8}$$

For sufficiently small ϵ, this difference will be as small as we please. Bearing in mind, that $0 < b < 1$, it is obvious that, as $\epsilon \to 0$ the second term on the right-hand side of (8) has zero limit. On the other hand,

$$\lim_{\epsilon \to 0} (I_2 - I_1) = \lim_{\epsilon \to 0} R_2(\epsilon) = \frac{1}{1-b} \lim_{\epsilon \to 0} \frac{\epsilon}{\left[f(x_1+\epsilon)\right]^b} =$$

$$= \frac{1}{1-b} \lim_{\epsilon \to 0} \left\{ \frac{\epsilon}{f(x_1+\epsilon) - f(x_1)} \left[f(x_1+\epsilon)\right]^{1-b} \right\} =$$

$$= \frac{1}{1-b} \, \frac{1}{f'(x_1)} \lim_{\epsilon \to 0} \left[f(x_1+\epsilon)\right]^{1-b} = 0 \, .$$

It is obvious from the statement of the theorem that this latter limit is zero since the function $f(x)$ is also continuous at $x = x_1$.

The theorem is proved.

Remark 1. If integral (1) is approximated by the arithmetic mean of the integrals (6) and (7), then the error bound $E(\epsilon)$ is

$$E(\epsilon) = \frac{1}{2} \left| R_2(\epsilon) - R_1(\epsilon) \right| .$$

That is

$$\left| I - \frac{I_1 + I_2}{2} \right| < E(\epsilon) \, .$$

Remark 2. This theorem can simply be extended to the case where the integrand of (1) has a singularity at $x = x_2$. Here we restricted our discussion to integrands with singularities at the end points of the interval. However, by splitting the integral into two integrals, singularities in the interior of the interval can also be handled.

REFERENCE

1. RALSTON, A.: *A First Course in Numerical Analysis.* Mc Graw-Hill, New York, 1965.

ÜBER DIE NÄHERUNGSWEISE BERECHNUNG EINES UNEIGENTLICHEN INTEGRALS

von

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Zusammenfassung

In dieser Arbeit ist ein Lehrsatz im endlichen Intervall für die Berechnung uneigentliches Integrals einer Funktion dargestellt, welche in irgendwelchem Endpunkt dieses Intervalls unendlich ist.

ПРИБЛИЖЕННОЕ ИСЧИСЛЕНИЕ НЕСОБСТВЕННЫХ ИНТЕГРАЛОВ

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Резюме

В настоящей работе излагается теорема, подходящая для вычисления с произвольной точностью несобственных интегралов функций, неограниченных в каком-нибудь конечном пункте ограниченного промежутка интегрирования.