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BOUNDARY INTEGRAL EQUATIONS FOR PLANE PROBLEMS IN TERMS OF STRESS FUNCTIONS OF ORDER ONE

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Abstract. The present paper is devoted to the plane problem of elastostatics assuming that the governing equations are given in terms of stress functions of order one. After clarifying the conditions of single valuedness we have constructed the fundamental solution for the dual basic equations. Then the integral equations of the direct method have been established. Numerical examples illustrate the applicability of the integral equations.

Mathematical Subject Classification: 74B05, 45A05
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1. Introduction, Preliminaries

In spite of a great number of publications devoted to plane problems there are only a few dealing with plane problems in terms of stress functions of order one. As regards classical elasticity we refer to the paper [1] and the book [2] by Jaswon and Smith in which the unknown biharmonic function (stress function of order two) is given in terms of two harmonic functions as a single layer potential and the authors set up a pair of integral equations for the unknown source densities.

Application of stress functions of order one was initiated by Frejis de Veubeke in a new complementary energy based finite element procedure [3,4] since the use of C^0 continuous stress functions of order one guarantees continuous surface tractions and makes possible to construct isoparametric elements. Further applications with an emphasis on three dimensional problems and laminated structures are due to Bertóti – see [5,6].

If one uses stress functions of order one calculation of stresses requires determination of first derivatives (in contrast to stress functions of order two from which stresses can be obtained in terms of second derivatives) and this property makes them attractive in boundary element applications though a further equation is needed to ensure that the stresses be symmetric.

As regards the derivation of integral equations for plane problems it is worth citing the papers by Heise [7,8], in which altogether 32 + 16 different integral equations are obtained with the aid of the singularity method. The reader taking an interest in the various formulations made by Heise is referred to these works and the references listed therein.

In the present paper we confine ourselves to the direct formulation within the framework of classical elasticity. Our aims are as follows:

1. Clarifying the conditions of single valuedness for a class of mixed boundary value problems assuming multiply connected regions.
2. Derivation of the fundamental solutions for the stress functions of order one.
3. Setting up the dual Somigliana relations (both for inner regions and for outer ones) from which the boundary integral equations of the direct method can be derived.
4. Presentation of some results obtained by solving the integral equations of the direct method.

We remark that some results of the paper can be found in the work [9].

2. Dual equations in terms of stress functions of order one

Throughout this paper $x_1 = x$ and $x_2 = y$ are rectangular Cartesian coordinates, referred to an origin O. The totality of $x_1 = x$ and $x_2 = y$ is denoted by x. {Greek}[Latin subscripts] are assumed to have the range {(1,2)}[(1,2,3)], summation over repeated subscripts is implied. The triple connected region under consideration is denoted by A_i – inner region – and is bounded by the outer contour

$$\mathcal{L}_0 = \mathcal{L}_{t1} \cup \mathcal{L}_{u2} \cup \mathcal{L}_{t3} \cup \mathcal{L}_{u4}$$

and the inner contours which – partly or wholly – consist of the arcs $\mathcal{L}_{t1}$, $\mathcal{L}_{t3}$, $\mathcal{L}_{t5}$ and $\mathcal{L}_{u2}$, $\mathcal{L}_{u4}$, $\mathcal{L}_{u6}$.

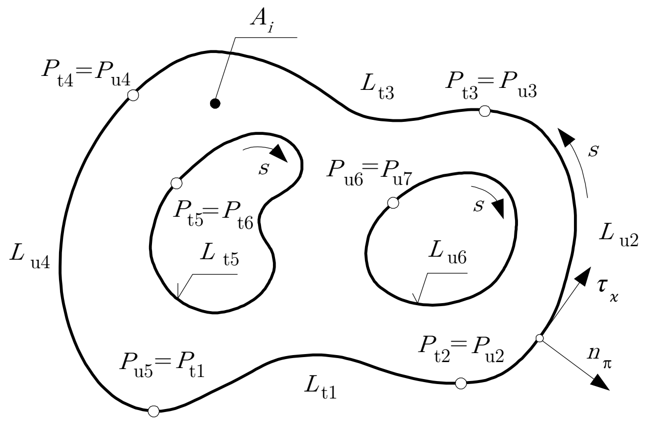

Figure 1

Further the inner contours $\mathcal{L}_1$ and $\mathcal{L}_2$ lie wholly in the interior of the outer contour $\mathcal{L}_0$ and they have no points in common. We stipulate that each contour has a continuously turning unit tangent τ_κ and admits a nonsingular parametrization in terms of its arc length s. The outer normal is denoted by n_π. In accordance with the notations introduced $\delta_{\kappa\lambda}$ is the Kronecker symbol, ∂_α stands for the derivatives with respect to x_α and $\epsilon_{3\kappa\lambda}$ is the permutation symbol. The {symmetric}[skew] part of a tensor, say the tensor $t_{\kappa\lambda}$, is denoted by $\{t_{(\kappa\lambda)}\}[t_{[\kappa\lambda]}]$.

Assuming plane strain, let u_κ, $e_{\kappa\lambda}$ and $t_{\kappa\lambda}$ be the displacement field and the in plane components of stress and strain, respectively. The stress functions of order one are denoted by $\mathcal{F}_\rho$.

For homogenous and isotropic material the plane strain problem of classical elasticity in dual system is governed by the dual kinematic equations

$$t_{\kappa\lambda} = \epsilon_{\kappa\rho 3}\mathcal{F}_\lambda\partial_\rho + \overset{o}{t}_{\kappa\lambda} \qquad x \in A_i \tag{2.1}$$

($\overset{o}{t}_{\kappa\lambda}$ is the particular solution that belongs to non-zero body forces), the inverse form of Hook's law

$$e_{\kappa\lambda} = \frac{1}{2\mu}\left(t_{(\kappa\lambda)} - \nu t_{\psi\psi}\delta_{\kappa\lambda}\right) \qquad x \in A_i \tag{2.2}$$

(μ is the shear modulus of elasticity, ν is the Poisson number), the dual balance equations

$$\epsilon_{\kappa\rho 3}e_{\lambda\kappa}\partial_\rho + \varphi_3\partial_\lambda = \epsilon_{\kappa\rho 3}\left(e_{\lambda\kappa} - \epsilon_{\lambda\kappa 3}\varphi_3\right)\partial_\rho = 0 \qquad x \in A_i \tag{2.3}$$

(equations of compatibility for a simply connected region; φ_3 is the rigid body rotation) and the symmetry condition

$$\epsilon_{3\kappa\lambda}t_{\kappa\lambda} = 0 \qquad x \in A_i \tag{2.4}$$

(equation of rotational equilibrium). If this equation is fulfilled, then one of the equations (2.2) can be omitted. In this way we have nine equations for the nine unknowns $\mathcal{F}_1$, $\mathcal{F}_2$, t_{11}, $t_{12} = t_{21}$, t_{22}, e_{11}, $e_{12} = e_{21}$, e_{22} and φ_3.

The field equations (2.1), (2.2), (2.3) and (2.4) should be associated with appropriate boundary conditions. If a contour is not divided into parts, then either tractions or displacements are imposed on it. If a contour is divided, then it is assumed to consists of arcs of even number on which displacements and tractions are imposed alternately. In the present case {tractions}[displacements] are given on the arc $\{\mathcal{L}_t = \mathcal{L}_{t1} \cup \mathcal{L}_{t3} \cup \mathcal{L}_{t5}\}[\mathcal{L}_u = \mathcal{L}_{u2} \cup \mathcal{L}_{u4} \cup \mathcal{L}_{u6}]$. We remark that hatted letters stand for the prescribed values.

Upon substitution of the equation (2.1) into the traction boundary condition $n_\pi t_{\pi\rho} = \hat{t}_\rho$ we arrive at the differential equation

$$\hat{t}_\rho - \overset{o}{t}_\rho = n_\kappa \epsilon_{\kappa\nu 3}\mathcal{F}_\rho\partial_\nu = \frac{d\mathcal{F}_\rho}{ds} \tag{2.5}$$

where $\overset{o}{t}_\rho = n_\pi \overset{o}{t}_{\pi\rho}$. One can readily check that the solution on the arcs of $\mathcal{L}_t$ assumes the form

$$\hat{\mathcal{F}}_\rho(s) = \int_{P_{ti}}^{s}\left[\hat{t}_\rho(\sigma) - \overset{o}{t}_\rho(\sigma)\right]d\sigma\ , \qquad s \in \mathcal{L}_{ti}\ , \qquad i = 1, 3, 5\ .$$

Let the constants $\underset{(ti)}{C}{}_\rho$ be that of integration. The condition

$$\mathcal{F}_\rho(s) = \hat{\mathcal{F}}_\rho(s) + \underset{(ti)}{C}{}_\rho \qquad i = 1, 3, 5 \tag{2.6}$$

is equivalent to the boundary condition (2.5) and conversely.

REMARK 1.: Observe that the number of undetermined constants of integration is two times as much as the number of those arcs on which tractions are imposed.

Since the displacements do not belong to the unknowns of the dual system one has to clarify what boundary conditions can be prescribed on the arcs constituting $\mathcal{L}_u$. Let

$$\mathcal{K} = \mathcal{K}(t_{\kappa\lambda}, \varphi_3) = -\frac{1}{2}\int_{A_i} t_{\kappa\lambda} e_{\kappa\lambda}\, dA + \int_{\mathcal{L}_u} n_\kappa t_{\kappa\lambda} \hat{u}_\lambda\, ds - \int_{A_i} t_{\kappa\lambda}\epsilon_{\kappa\lambda 3}\varphi_3 dA \tag{2.7}$$

be a modified form of the complementary energy functional. (The modification is a must in order to keep up the rotational equilibrium.) Solution to the problem posed can be sought by making use of the stationary condition

$$\delta\mathcal{K} = 0 \tag{2.8}$$

since the latter equation should ensure all the conditions the strains $e_{\kappa\lambda}$ and the rigid body rotation φ_3 are to meet in order to be kinematically admissible. In the functional (2.7) $e_{\kappa\lambda}$ is given in terms of the stresses $t_{\kappa\lambda}$ via Hook's law while the stresses $t_{\kappa\lambda}$ should satisfy the equilibrium equation and the traction boundary condition though it is not necessary for them to be symmetric. Consequently, the variations of stresses can not be arbitrary but should meet the conditions

$$\delta t_{\kappa\lambda}\partial_\kappa = 0 \qquad x \in A \qquad \text{and} \qquad n_\kappa \delta t_{\kappa\lambda} = 0 \qquad x \in \mathcal{L}_t\ . \tag{2.9}$$

Both conditions are satisfied if $\delta t_{\kappa\lambda}$ is given in terms of the variations of stress functions

$$\delta t_{\kappa\lambda} = \epsilon_{\kappa\rho 3}\delta\mathcal{F}_\lambda \partial_\rho \tag{2.10}$$

where $\delta\mathcal{F}_\lambda$ is arbitrary on A_i. However, with regard to (2.6) it follows that on $\mathcal{L}_t$

$$\delta\mathcal{F}_\rho(s) = \delta \underset{(ti)}{C}{}_\rho \qquad (i = 1, 3, 5)\ . \tag{2.11}$$

Derivation of the conditions the strains $e_{\kappa\lambda}$ and the rigid body rotation should meet in order to be kinematically admissible requires the transformation of the stationary condition

$$\delta\mathcal{K} = -\int_{A_i} e_{\kappa\lambda}\, \delta t_{\kappa\lambda}\, dA + \int_{\mathcal{L}_u} n_\kappa \delta t_{\kappa\lambda}\hat{u}_\lambda\, ds - \int_{A_i} \delta t_{\kappa\lambda}\epsilon_{\kappa\lambda 3}\varphi^3 dA - \int_{A_i} t_{\kappa\lambda}\epsilon_{\kappa\lambda 3}\delta\varphi^3 dA = 0\,. \tag{2.12}$$

The main steps of the transformations are as follows:

1. Substitution of the condition (2.10) into the first and second surface integrals and substitution of (2.1) into the third surface integral.

2. Substitution of the relation

$$n_\kappa \epsilon_{\kappa\rho 3} \delta \mathcal{F}_\lambda \partial_\rho = \frac{d\delta \mathcal{F}_\lambda}{ds}$$

   into the line integral taken on $\mathcal{L}_u$.
3. Application of the Green-Gauss theorem [10] to the first and second surface integrals.
4. Performation of partial integrations on the arcs constituting $\mathcal{L}_u$ taking into account the validity of (2.11) at the extremities of the arcs.
5. Division of the line integrals obtained by the application of the Green-Gauss theorem by using the relation

$$\int_{\mathcal{L}} .... = \int_{\mathcal{L}_u} .... + \int_{\mathcal{L}_t} ....$$

   then substitution of (2.11) into the line integrals taken on $\mathcal{L}_t$.
6. Transformation of the result making use of the equation

$$n_\rho \epsilon_{\kappa\rho 3} = -\tau_\kappa \ . \tag{2.13}$$

After performing the steps listed above

$$\begin{aligned}\delta K = &\int_{A_i} (\epsilon_{\kappa\rho 3} e_{\kappa\lambda} \partial_\rho + \varphi_3 \partial_\lambda)\, \delta \mathcal{F}_\lambda dA - \int_{A_i} (\mathcal{F}_\psi \partial_\psi)\, \delta\varphi_3 dA \\ &+ \sum_{i=2,4,6} \int_{\mathcal{L}_{ui}} \left\{ n_\pi [\epsilon_{\pi\kappa 3} e_{\kappa\lambda} - \delta_{\pi\lambda}\varphi_3] - \frac{d\hat{u}_\lambda}{ds} \right\} \delta \mathcal{F}_\lambda ds \\ &+ \sum_{i=1,3,5} \left\{ \int_{\mathcal{L}_{ti}} n_\pi [\epsilon_{\pi\kappa 3} e_{\kappa\lambda} - \delta_{\pi\lambda}\varphi_3] ds - \hat{u}_\lambda |_{P_{ti}}^{P_{t,i+1}} \right\} \delta \underset{(ti)}{C}{}_\lambda = 0\end{aligned}$$

is the stationary condition. Since the variations are arbitrary from this condition it follows the compatibility condition (2.3), the symmetry condition (2.4) – in the latter $t_{\kappa\lambda}$ is given in terms of $\mathcal{F}_\psi$ –, the strain boundary condition

$$\frac{d\hat{u}_\lambda}{ds} = n_\pi [\epsilon_{\pi\kappa 3} e_{\kappa\lambda} - \delta_{\pi\lambda}\varphi_3] \ , \tag{2.14}$$

the compatibility condition in the large

$$\int_{\mathcal{L}_{t5}} n_\pi [\epsilon_{\pi\kappa 3} e_{\kappa\lambda} - \delta_{\pi\lambda}\varphi_3] ds = 0 \tag{2.15}$$

and the supplementary condition of single valuedness

$$\int_{\mathcal{L}_{ti}} n_\pi [\epsilon_{\pi\kappa 3} e_{\kappa\lambda} - \delta_{\pi\lambda}\varphi_3] ds - \hat{u}_\lambda |_{P_{ti}}^{P_{t,i+1}} = 0 \qquad (i = 1, 3) \ . \tag{2.16}$$

REMARK 2.: The strain boundary condition can also be obtained if one regards the primal kinematic equation

$$e_{\kappa\lambda} = \frac{1}{2}(u_\kappa \partial_\lambda + u_\lambda \partial_\kappa)$$

on the contour, multiplies it by $n_\pi \epsilon_{\pi\kappa 3} = \tau_\kappa$ taking into account that $u_{[\kappa}\partial_{\lambda]} = -\epsilon_{\kappa\lambda 3}\varphi_3$.

REMARK 3.: Both the compatibility condition in the large (2.15) and the supplementary condition of single valuedness (2.16) can be set up by integrating the strain boundary condition (2.14) appropriately.

REMARK 4.: It can be shown that only two of the three conditions (as a matter of fact three times two conditions) (2.15) and (2.16) are independent of each other. In accordance with this, one can set one times two of the three times two undetermined constants of integration $\underset{(ti)}{C}_\rho$, say $\underset{(t1)}{C}_\rho$, to zero since there belong no stresses to the stress function $\mathcal{F}_\rho = \underset{(t1)}{C}_\rho$ = constant. In other words, we have as many independent macro conditions of single valuedness as there are undetermined constants of integration.

3. Basic equations and fundamental solutions

Here and in the sequel we shall assume that there are no body forces. Substituting the dual kinematic equation (2.1) into Hook's law (2.2) and the result into the compatibility equations (2.3) we have

$$\frac{1}{2\mu}(1-\nu)\Delta\mathcal{F}_1 - \frac{1}{2\mu}(\frac{1}{2}-\nu)(\mathcal{F}_1\partial_1 + \mathcal{F}_2\partial_2)\partial_1 + \varphi_3\partial_1 = 0\,, \tag{3.1a}$$

$$\frac{1}{2\mu}(1-\nu)\Delta\mathcal{F}_2 - \frac{1}{2\mu}(\frac{1}{2}-\nu)(\mathcal{F}_1\partial_1 + \mathcal{F}_2\partial_2)\partial_2 + \varphi_3\partial_2 = 0\,. \tag{3.1b}$$

These equations are associated with the symmetry condition in terms of $\mathcal{F}_\rho$:

$$\mathcal{F}_1\partial_1 + \mathcal{F}_2\partial_2 = 0\,. \tag{3.1c}$$

Upon substitution of (3.1c) into (3.1a,b) the latter, two equations become much simpler. In spite of that and for the sake of a comparison with the plane orthotropic case, the work on that problem is in progress, we do not change the above equations. Introducing the notations

$$[\mathfrak{D}_{lk}] = \begin{bmatrix} \frac{1}{2\mu}(1-\nu)\Delta - \frac{1}{2\mu}(\frac{1}{2}-\nu)\partial_1\partial_1 & -\frac{1}{2\mu}(\frac{1}{2}-\nu)\partial_1\partial_2 & -\partial_1 \\ -\frac{1}{2\mu}(\frac{1}{2}-\nu)\partial_2\partial_1 & \frac{1}{2\mu}(1-\nu)\Delta - \frac{1}{2\mu}(\frac{1}{2}-\nu)\partial_2\partial_2 & -\partial_2 \\ -\partial_1 & -\partial_2 & 0 \end{bmatrix} \tag{3.2a}$$

and

$$\mathfrak{u}_k = (\mathcal{F}_1, \mathcal{F}_2, -\varphi_3) \tag{3.2b}$$

the basic equation takes the form

$$\mathfrak{D}_{ik}\mathfrak{u}_k = 0\,. \tag{3.3}$$

Let D_{kj} be the cofactor of $\mathfrak{D}_{jk}$:

$$[D_{kl}] = \begin{bmatrix} -\partial_2\partial_2 & \partial_1\partial_2 & \dfrac{1}{2\mu}(1-\nu)\Delta\partial_1 \\ \partial_2\partial_1 & -\partial_1\partial_1 & \dfrac{1}{2\mu}(1-\nu)\Delta\partial_2 \\ \dfrac{1}{2\mu}(1-\nu)\Delta\partial_1 & \dfrac{1}{2\mu}(1-\nu)\Delta\partial_2 & \dfrac{1}{8\mu^2}(1-\nu)\Delta\Delta \end{bmatrix}. \tag{3.4}$$

It is obvious that

$$D_{ik}\mathfrak{D}_{kl} = \mathfrak{D}_{ik}D_{kl} = \det(\mathfrak{D}_{jl})\,\delta_{kl} \tag{3.5}$$

where

$$\det(\mathfrak{D}_{jl}) = -\frac{1}{2\mu}(1-\nu)\Delta\Delta\,. \tag{3.6}$$

If we introduce a new unknown χ_l [11], [12] defined by the equation

$$\mathfrak{u}_k = D_{kl}\chi_l \tag{3.7}$$

and substitute it back into the equation (3.3) we have an uncoupled system of differential equations

$$\mathfrak{D}_{ik}\mathfrak{u}_k = \mathfrak{D}_{ik}D_{kl}\chi_l = \det(\mathfrak{D}_{jl})\chi_i = 0\,. \tag{3.8}$$

Let $Q(\xi_1,\xi_2)$ and $M\,(x_1,x_2)$ be two points in the plane of strain (the source point and the point of effect). Further let $\mathbf{e}$ with components e_i be a unit vector at Q. We shall assume temporarily that the point Q is fixed. The distance between Q and M is R, the position vector of M relative to Q is r_κ. Solution to the differential equation

$$\mathfrak{D}_{ik}\mathfrak{u}_k + \delta(M-Q)e_i = 0$$

is referred to as fundamental solution. It is clear from all that has been said – see (3.7), (3.8) and (3.5) – that the fundamental solution is obtainable from the fundamental solution for the Galorkin functions, i.e., from the solution of the differential equation

$$\det(\mathfrak{D}_{jl})\chi_i + \delta(M-Q)e_i = -\frac{1}{2\mu}(1-\nu)\Delta\Delta\chi_i + \delta(M-Q)e_i = 0\,. \tag{3.9}$$

Making use of the fundamental solution

$$\chi_i(M,Q) = \frac{\mu}{4\pi(1-\nu)}R^2(\ln R - 1)e_i \tag{3.10}$$

valid for the plane biharmonic equation [2] we have

$$\mathfrak{u}_k = \mathfrak{U}_{kl}(M,Q)e_l(Q) \tag{3.11}$$

where

$$[\mathfrak{U}_{kl}(M,Q)] = \frac{\mu}{4\pi(1-\nu)}\begin{bmatrix} -2\ln R - 3 - 2\dfrac{r_2r_2}{R^2} & 2\dfrac{r_1r_2}{R^2} & \dfrac{2}{\mu}(1-\nu)\dfrac{r_1}{R^2} \\ 2\dfrac{r_2r_1}{R^2} & -2\ln R - 3 - 2\dfrac{r_1r_1}{R^2} & \dfrac{2}{\mu}(1-\nu)\dfrac{r_2}{R^2} \\ \dfrac{2}{\mu}(1-\nu)\dfrac{r_1}{R^2} & \dfrac{2}{\mu}(1-\nu)\dfrac{r_2}{R^2} & 0 \end{bmatrix}. \tag{3.12}$$

Remark 5.: The fundamental solution $\mathfrak{U}_{kl}(M,Q)$ satisfies the symmetry conditions

$$\mathfrak{U}_{kl}(M,Q) = \mathfrak{U}_{lk}(M,Q) = \mathfrak{U}_{kl}(Q,M) = \mathfrak{U}_{lk}(Q,M)\ . \tag{3.13}$$

Consequently

$$\mathfrak{u}_k = \mathfrak{U}_{kl}(M,Q)e_l(Q) = e_l(Q)\mathfrak{U}_{lk}(M,Q)\ . \tag{3.14}$$

Remark 6.: Each row and column of $\mathfrak{U}_{kl}(M,Q)$ as a three dimensional vector satisfies the basic equation (3.3) both in M and in Q.

Substituting the columns of $\mathfrak{U}_{kl}(M,Q)$ into (2.1) and recalling that the particular solution for stresses is assumed to be zero, we have the fundamental solution for stresses

$$\begin{bmatrix} t_{11} \\ t_{12} \\ t_{22} \end{bmatrix} = \frac{\mu}{4\pi(1-\nu)R^2} \begin{bmatrix} -6r_2 + 4\dfrac{r_2^3}{R^2} & 2r_1 - \dfrac{4r_1r_2^2}{R^2} & -\dfrac{4}{\mu}(1-\nu)\dfrac{r_1r_2}{R^2} \\ 2r_1 - \dfrac{4r_2^2r_1}{R^2} & -2r_2 + \dfrac{4r_1^2r_2}{R^2} & \dfrac{2}{\mu}(1-\nu)\dfrac{r_1^2-r_2^2}{R^2} \\ -2r_2 + \dfrac{4r_1^2r_2}{R^2} & 6r_1 - 4\dfrac{r_1^3}{R^2} & \dfrac{4}{\mu}(1-\nu)\dfrac{r_1r_2}{R^2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{3.15}$$

It can be shown that $t_{12} = t_{21}$.

With the aid of Hook's law (2.2) one obtains the fundamental solution for strains

$$\begin{bmatrix} e_{11} \\ e_{12} \\ e_{22} \end{bmatrix} = \frac{\hat{C}}{R^2} \begin{bmatrix} -2(3-2v)r_2 + 4\dfrac{r_2^3}{R^2} & 2(1-2v)r_1 - 4\dfrac{r_1r_2^2}{R^2} & -\dfrac{4}{\mu}(1-\nu)\dfrac{r_1r_2}{R^2} \\ 2r_1 - \dfrac{4r_2^2r_1}{R^2} & -2r_2 + \dfrac{4r_1^2r_2}{R^2} & \dfrac{2}{\mu}(1-\nu)\dfrac{r_1^2-r_2^2}{R^2} \\ -2(1-2v)r_2 + 4\dfrac{r_1^2r_2}{R^2} & 2(3-2v)r_1 - 4\dfrac{r_1^3}{R^2} & \dfrac{4}{\mu}(1-\nu)\dfrac{r_1r_2}{R^2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{3.16a}$$

$$\hat{C} = \frac{1}{8\pi(1-\nu)}\ . \tag{3.16b}$$

For our later considerations we shall introduce the notation

$$\mathfrak{t}_\lambda = -\frac{du_\lambda}{ds} \tag{3.17}$$

where the vector $\mathfrak{t}_\lambda$ is referred to as displacement derivative. Comparing (2.14), (3.16a,b) and (3.17) for the vector $\mathfrak{t}_\lambda$ from the fundamental solution we get

$$\mathfrak{t}_\lambda(\overset{o}{M}) = e_l(Q)\mathfrak{T}_{l\lambda}(\overset{o}{M},Q) \tag{3.18a}$$

where

$$\mathfrak{T}_{l\lambda}(\overset{o}{M},Q)=\frac{\hat{C}}{R^2}\begin{bmatrix} \begin{matrix} n_1r_1\left(4\dfrac{r_2^2}{R^2}-2(3-2v)\right) \\ +n_2r_2\left(4\dfrac{r_2^2}{R^2}-2(3-2v)\right)\end{matrix} & \begin{matrix} -n_2r_1\left(4\dfrac{r_2^2}{R^2}+2(1-2v)\right) \\ -n_1r_2\left(4\dfrac{r_1^2}{R^2}-2(1-2v)\right)\end{matrix} \\ \begin{matrix} -n_1r_2\left(4\dfrac{r_1^2}{R^2}+2(1-2v)\right) \\ -n_2r_1\left(4\dfrac{r_2^2}{R^2}-2(1-2v)\right)\end{matrix} & \begin{matrix} n_2r_2\left(4\dfrac{r_1^2}{R^2}-2(3-2v)\right) \\ +n_1r_1\left(4\dfrac{r_1^2}{R^2}-2(3-2v)\right)\end{matrix} \\ \begin{matrix} -n_1\dfrac{2}{\mu}(1-\nu)\dfrac{r_1^2-r_2^2}{R^2} \\ -n_2\dfrac{4}{\mu}(1-\nu)\dfrac{r_1r_2}{R^2}\end{matrix} & \begin{matrix} -n_1\dfrac{4}{\mu}(1-\nu)\dfrac{r_1r_2}{R^2} \\ +n_2\dfrac{2}{\mu}(1-\nu)\dfrac{r_1^2-r_2^2}{R^2}\end{matrix} \end{bmatrix}. \tag{3.18b}$$

Here and in the sequel the small circle over the letters M and/or Q has the meaning that the corresponding point is located on the contour. The normal n_λ is taken at the point $\overset{o}{M}$.

Remark 7.: Recalling that in the circle of the boundary value problems considered either the stress functions or the derivative of the displacements with respect to the arc coordinate can be prescribed at a point on the contour for our later consideration, it is worth giving the value of the stress functions from the fundamental solution on the boundary:

$$\mathfrak{u}_\lambda = e_l(Q)\mathfrak{U}_{l\lambda}(\overset{o}{M},Q)\;. \tag{3.19}$$

The displacement derivative from the fundamental solution is given by (3.18a,b).

4. Somigliana identity and formulae in dual system – inner region

Here and in the sequel it is assumed that the region A_i under consideration is simply connected and lies wholly in finite. The contour $\mathcal{L}_0$ is divided into arcs of even number on which displacements (or their derivatives with respect to s) and tractions (or stress functions) can be imposed alternately. In Figure 2 the region A_i is divided into four arcs though this fact does not play any role in the transformations.

The functions $\mathcal{F}_\psi$, $t_{\kappa\lambda}$, $e_{\kappa\lambda}$ and φ_3 are referred to as an elastic state of the region A_i provided that they satisfy the field equations (2.1), (2.2), (2.3) and (2.4). Let

$$\mathcal{F}_\psi,\ t_{\kappa\lambda},\ e_{\kappa\lambda},\ \varphi_3 \qquad \text{and} \qquad \overset{*}{\mathcal{F}}_\psi,\ \overset{*}{t}_{\kappa\lambda},\ \overset{*}{e}_{\kappa\lambda},\ \overset{*}{\varphi}_3$$

be two elastic states of the region A_i. Applying the Green–Gauss theorem and taking

(2.1) into account (since there are no body forces $\overset{o}{t}_{\kappa\lambda} = 0$) one can write

$$\int_{A_i} [\epsilon_{\kappa\rho 3} e_{\kappa\lambda} \partial_\rho + \varphi_3 \partial_\lambda] \overset{*}{\mathcal{F}}_\lambda dA - \int_{A_i} (\mathcal{F}_\psi \partial_\psi) \overset{*}{\varphi}_3 dA = \tag{4.1}$$

$$= \oint_{\mathcal{L}_o} n_\pi [\epsilon_{\pi\kappa 3} e_{\kappa\lambda} - \delta_{\pi\lambda} \varphi_3] \overset{*}{\mathcal{F}}_\lambda ds$$

$$- \int_{A_i} \left(\overset{*}{\mathcal{F}}_\psi \partial_\psi \right) \varphi_3 dA - \int_{A_i} (\mathcal{F}_\psi \partial_\psi) \overset{*}{\varphi}_3 dA \, .$$

Remark 7.: Observe that the first surface integral and the sum of the last two surface integrals on the right side do not depend on the placement of the asterisk which can be put over the first or the second factor of the corresponding products.

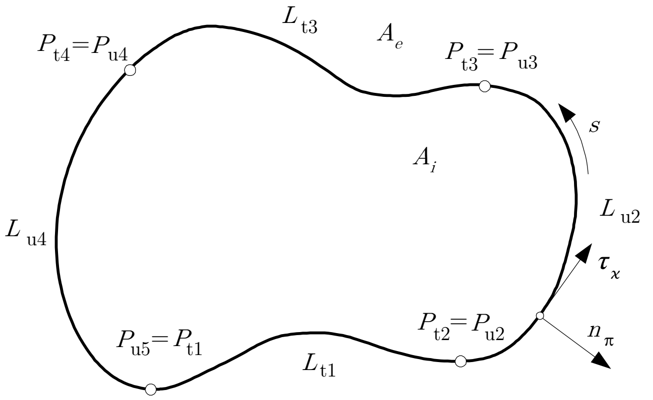

Figure 2

If we replace the asterisk over the letters denoting the first elastic state and subtract (4.1) from the resulting equation, then we get the dual Somigliana identity for plane problems

$$\int_{A_i} [\epsilon_{\kappa\rho 3} \overset{*}{e}_{\kappa\lambda} \partial_\rho + \overset{*}{\varphi}_3 \partial_\lambda] \mathcal{F}_\lambda dA - \int_{A_i} \left(\overset{*}{\mathcal{F}}_\psi \partial_\psi \right) \varphi_3 dA \tag{4.2}$$

$$- \int_{A_i} [\epsilon_{\kappa\rho 3} e_{\kappa\lambda} \partial_\rho + \varphi_3 \partial_\lambda] \overset{*}{\mathcal{F}}_\lambda dA - \int_{A_i} (\mathcal{F}_\psi \partial_\psi) \overset{*}{\varphi}_3 dA =$$

$$= \oint_{\mathcal{L}_o} n_\pi [\epsilon_{\pi\kappa 3} \overset{*}{e}_{\kappa\lambda} - \delta_{\pi\lambda} \overset{*}{\varphi}_3] \mathcal{F}_\lambda ds - \oint_{\mathcal{L}_o} n_\pi [\epsilon_{\pi\kappa 3} e_{\kappa\lambda} - \delta_{\pi\lambda} \varphi_3] \overset{*}{\mathcal{F}}_\lambda ds.$$

On the left side we have the integrals of the basic equations. As regards the right side, we have the integrals of those quantities one can prescribe on the boundary. Recalling the relations (3.1a,b,c), (3.2a,b), (3.3) giving the basic equations and the notation (3.17) in which the derivative is given by (2.14), one can cast the Somigliana identity into a form similar to the Green identity [2]

$$\int_{A_i} \left[\mathfrak{u}_k \left(\mathfrak{D}_{kl} \overset{*}{\mathfrak{u}}_l \right) - \overset{*}{\mathfrak{u}}_k \left(\mathfrak{D}_{kl} \mathfrak{u}_l \right) \right] dA = \oint_{\mathcal{L}_o} [\mathfrak{u}_\lambda \overset{*}{\mathfrak{t}}_\lambda - \overset{*}{\mathfrak{u}}_\lambda \mathfrak{t}_\lambda] \, ds \, . \tag{4.3}$$

REMARK 8.: When deriving (4.3) we have never taken into consideration that $\overset{*}{\mathfrak{u}}_k$ and $\mathfrak{u}_k$ are compatible, i.e., fulfill the basic equation (3.3). Consequently, the equation (4.3) is really an identity which is always valid if $\overset{*}{\mathfrak{u}}_k$ and $\mathfrak{u}_k$ are differentiable as many times as required – in other respects both functions can be arbitrary.

In order to establish the dual Somigliana relations it is assumed that $\overset{*}{\mathfrak{u}}_k$ is an elastic state given by (3.11) and (3.12). Quantities in the line integrals are defined by (3.19) and (3.18a,b).

In what follows we shall utilize that $\mathfrak{u}_k$ is also an elastic state.

Since the state identified by the asterisk is singular at the point Q (at the source point), we distinguish three cases depending on the location of Q with respect to the region A_i.

1. If $Q \in A_i$, then the neighborhood of Q with radius R_ε, which is denoted by A_ε and is assumed to lie in A_i, is removed from A_i and we apply the dual Somigliana identity to the double connected domain $A' = A_i \setminus A_\varepsilon$. We remark that the contour $\mathcal{L}_\varepsilon$ of A_ε and the arc $\mathcal{L}_\varepsilon^{'}$ of the contour $\mathcal{L}_\varepsilon$ within A_i coincide with each other.
2. If $Q = \overset{o}{Q} \in \partial A_i = \mathcal{L}_o$, then the part $A_i \cap A_\varepsilon$ of the neighborhood A_ε of Q is removed from A_i and we apply the dual Somigliana identity to the simply connected region $A' = A_i \setminus (A_i \cap A_\varepsilon)$. If this is the case, the contour of the simply connected region consists of two arcs, the arc $\mathcal{L}_o^{'}$ left from $\mathcal{L}_o$ after the removal of A_ε and the arc $\mathcal{L}_\varepsilon^{'}$, i.e., the part of $\mathcal{L}_\varepsilon$ that lies within A_i.
3. If $Q \notin (A_i \cup \mathcal{L}_o)$, we apply the Somigliana identity to the original region A_i.

Since both $\overset{*}{\mathfrak{u}}_k$ and $\mathfrak{u}_k$ are elastic states, the surface integrals in (4.3) are identically equal to zero. In what follows we regard the three cases one by one focusing attention on the line of thought.

1. Making use of all that has been said above for $Q \in A_i$, it follows from (4.3) that

$$\oint_{\mathcal{L}_o} [\mathfrak{T}_{k\lambda}(\overset{o}{M}, Q)\mathfrak{u}_\lambda(\overset{o}{M}) - \mathfrak{U}_{k\lambda}(\overset{o}{M}, Q)\mathfrak{t}_\lambda(\overset{o}{M})]\, ds_{\overset{o}{M}} + \oint_{\mathcal{L}_\varepsilon} [\mathfrak{T}_{k\lambda}(M, Q)\mathfrak{u}_\lambda(M) - \mathfrak{U}_{k\lambda}(M, Q)\mathfrak{t}_\lambda(M)]\, ds_M = 0\,. \tag{4.4}$$

It can be shown that

$$\oint_{\mathcal{L}_\varepsilon} \mathfrak{T}_{\kappa\lambda}(M, Q)\, ds_M = \delta_{\kappa\lambda}\,, \tag{4.5a}$$

$$\lim_{R_\varepsilon \to 0} \oint_{\mathcal{L}_\varepsilon} \mathfrak{T}_{\kappa\lambda}(M, Q)\, [\mathfrak{u}_\lambda(M) - \mathfrak{u}_\lambda(Q)]\; ds_M = 0\,, \tag{4.5b}$$

$$\oint_{\mathcal{L}_\varepsilon} \mathfrak{T}_{3\lambda}(M, Q)\, ds_M = \frac{1}{4\pi\mu}\frac{1}{R_\varepsilon}\int_0^{2\pi} \cos\varphi\, d\varphi = 0\,, \tag{4.5c}$$

(The latter equation is fulfilled for any $R_\varepsilon \neq 0$. Consequently, if $R_\varepsilon \to 0$ the limit of the integral is also zero.)

$$\lim_{R_\varepsilon\to 0}\oint_{\mathcal{L}_\varepsilon}\mathfrak{T}_{3\lambda}(M,Q)\left[\mathfrak{u}_\lambda(M)-\mathfrak{u}_\lambda(Q)\right]\,ds_M = \tag{4.5d}$$
$$=\lim_{R_\varepsilon\to 0}\left\{\oint_{\mathcal{L}_\varepsilon}\mathfrak{T}_{3\lambda}(M,Q)\left[\left.\frac{\partial\mathfrak{u}_\lambda}{\partial x_1}\right|_Q r_1+\left.\frac{\partial\mathfrak{u}_\lambda}{\partial x_2}\right|_Q r_2\right]\,ds_M+I_\varepsilon(R_\varepsilon)\right\}=$$
$$=\frac{1}{4\pi\mu}(t_{21}-t_{12})+\lim_{R_\varepsilon\to 0}I_\varepsilon(R_\varepsilon)=0\ ,$$

(Since $\mathfrak{u}_\lambda(M)$ is an elastic state, the stress tensor is symmetric and the expression I_ε is homogenous in R_ε.)

$$\lim_{R_\varepsilon\to 0}\oint_{\mathcal{L}_\varepsilon}\mathfrak{U}_{\kappa\lambda}(M,Q)\mathfrak{t}_\lambda(M)\,ds_M=0\ , \tag{4.5e}$$

(The relation

$$\mathfrak{t}_\lambda=-\frac{\partial u_\lambda}{\partial s}=-\frac{\partial u_\lambda}{\partial x_1}\frac{dx_1}{ds}-\frac{\partial u_\lambda}{\partial x_2}\frac{dx_2}{ds}$$

has been applied here and it should also be applied in the following transformation. In addition one should take the limit $\lim_{R_\varepsilon\to 0}R_\varepsilon\ln R_\varepsilon=0$ into consideration.)

$$\lim_{R_\varepsilon\to 0}\oint_{\mathcal{L}_\varepsilon}\mathfrak{U}_{3\lambda}(M,Q)\mathfrak{t}_\lambda(M)\,ds_M=\varphi_3|_Q=-\mathfrak{u}_3|_Q\ . \tag{4.5f}$$

If we take the limit of the equation (4.4) as $R_\varepsilon\to 0$ and substitute the formulae (4.5a,...,e), we obtain the first dual Somigliana relation:

$$\mathfrak{u}_k(Q)=\oint_{\mathcal{L}_o}\mathfrak{U}_{k\lambda}(\overset{o}{M},Q)\mathfrak{t}_\lambda(\overset{o}{M})\,ds_{\overset{o}{M}}-\oint_{\mathcal{L}_o}\mathfrak{T}_{k\lambda}(\overset{o}{M},Q)\mathfrak{u}_\lambda(\overset{o}{M})\,ds_{\overset{o}{M}}\ . \tag{4.6}$$

2. If $Q=\overset{o}{Q}\in\partial A=\mathcal{L}_o$, it follows from (4.3) by the steps leading to (4.4) that

$$\int_{\mathcal{L}'_o}[\mathfrak{T}_{\kappa\lambda}(\overset{o}{M},\overset{o}{Q})\mathfrak{u}_\lambda(\overset{o}{M})-\mathfrak{U}_{\kappa\lambda}(\overset{o}{M},\overset{o}{Q})\mathfrak{t}_\lambda(\overset{o}{M})]\,ds_{\overset{o}{M}} \tag{4.7}$$
$$+\int_{\mathcal{L}'_\varepsilon}[\mathfrak{T}_{\kappa\lambda}(M,\overset{o}{Q})\mathfrak{u}_\lambda(M)-\mathfrak{U}_{\kappa\lambda}(M,\overset{o}{Q})\mathfrak{t}_\lambda(M)]\,ds_M=0\,.$$

It can be shown that

$$\lim_{R_\varepsilon\to 0}\int_{\mathcal{L}'_\varepsilon}\mathfrak{T}_{\kappa\lambda}(M,\overset{o}{Q})\,ds_M=c_{\kappa\lambda}(\overset{o}{Q})\ , \tag{4.8a}$$

where $c_{\kappa\lambda}(\overset{o}{Q})=\delta_{\kappa\lambda}/2$ if the contour $\mathcal{L}_o$ is smooth at the point $\overset{o}{Q}$. If the contour is not smooth, then $c_{\kappa\lambda}(\overset{o}{Q})$ depends on the angle formed by the tangents to the contour at $\overset{o}{Q}$.

It can also be proved that

$$\lim_{R_\varepsilon \to 0} \int_{\mathcal{L}'_\varepsilon} \mathfrak{T}_{\kappa\lambda}(M, \overset{o}{Q}) \left[\mathfrak{u}_\lambda(M) - \mathfrak{u}_\lambda(\overset{o}{Q}) \right] ds_M = 0 \,, \tag{4.8b}$$

$$\lim_{R_\varepsilon \to 0} \int_{\mathcal{L}'_\varepsilon} \mathfrak{U}_{\kappa\lambda}(M, \overset{o}{Q}) \mathfrak{t}_\lambda(M) \, ds_M = 0 \,. \tag{4.8c}$$

If we take the limit of equation (4.7) as $R_\varepsilon \to 0$ and substitute the formulae (4.8a,b,c) we obtain the second dual Somigliana relation:

$$c_{\kappa\lambda}(\overset{o}{Q}) \mathfrak{u}_\lambda(\overset{o}{Q}) = \oint_{\mathcal{L}_o} \mathfrak{U}_{\kappa\lambda}(\overset{o}{M}, \overset{o}{Q}) \mathfrak{t}_\lambda(\overset{o}{M}) \, ds_{\overset{o}{M}} - \oint_{\mathcal{L}_o} \mathfrak{T}_{\kappa\lambda}(\overset{o}{M}, \overset{o}{Q}) \mathfrak{u}_\lambda(\overset{o}{M}) \, ds_{\overset{o}{M}} \,. \tag{4.9}$$

REMARK 9.: The two line integrals in (4.9) should be taken in principal value.

REMARK 10.: The integral equation (4.9) with unknowns $\mathfrak{u}_\lambda(\overset{o}{M})$ on $\mathcal{L}_u$ and $\mathfrak{t}_\lambda(\overset{o}{M})$ on $\mathcal{L}_t$ is that of the direct method.

3. If $Q \notin (A \cup \mathcal{L}_o)$, then the line integral in the identity (4.3) is taken on $\mathcal{L}_o$ (the surface integrals on the right side are ab ovo equal to zero) and by repeating the steps leading to (4.4) we have the third dual SOMIGLIANA formula:

$$0 = \oint_{\mathcal{L}_o} \mathfrak{U}_{k\lambda}(\overset{o}{M}, Q) \mathfrak{t}_\lambda(\overset{o}{M}) \, ds_{\overset{o}{M}} - \oint_{\mathcal{L}_o} \mathfrak{T}_{k\lambda}(\overset{o}{M}, Q) \mathfrak{u}_\lambda(\overset{o}{M}) \, ds_{\overset{o}{M}} \,. \tag{4.10}$$

Making use of the first dual Somigliana formula (4.6) and the dual kinematic equation (2.1) (in the latter case one has to recall that there are no body forces, consequently the particular solution is zero) one obtains the formula for the stresses $\mathfrak{s}_k = (t_{11}, t_{12}, t_{22})$ by performing the corresponding derivations

$$\mathfrak{s}_k(Q) = \oint_{\mathcal{L}_o} \mathrm{D}_{k\lambda}(\overset{o}{M}, Q) \mathfrak{t}_\lambda(\overset{o}{M}) \, ds_{\overset{o}{M}} - \oint_{\mathcal{L}_o} \mathrm{S}_{k\lambda}(\overset{o}{M}, Q) \mathfrak{u}_\lambda(\overset{o}{M}) \, ds_{\overset{o}{M}} \tag{4.11}$$

where the elements of $\mathrm{D}_{k\lambda}(\overset{o}{M}, Q)$ and $\mathrm{S}_{k\lambda}(\overset{o}{M}, Q)$ are given by

$$\mathrm{D}_{k\lambda}(\overset{o}{M}, Q) = \frac{\mu}{4\pi(1-\nu)R^2} \hat{\mathrm{D}}_{k\lambda}(\overset{o}{M}, Q) \,, \tag{4.12a}$$

$$\hat{\mathrm{D}}_{k\lambda}(\overset{o}{M}, Q) = \begin{bmatrix} 6r_2 - 4r_2^3/R^2 & -2r_1 + 4r_1 r_2^2/R^2 \\ -2r_1 + 4r_1 r_2^2/R^2 & 2r_2 - 4r_1^2 r_2/R^2 \\ 2r_2 - 4r_1^2 r_2/R^2 & -6r_1 + 4r_1^3/R^2 \end{bmatrix} \tag{4.12b}$$

and

$$\mathrm{S}_{k\lambda}(\overset{o}{M}, Q) = \frac{1}{8\pi(1-\nu)R^2} \hat{\mathrm{S}}_{k\lambda}(\overset{o}{M}, Q) \,, \tag{4.13a}$$

$$\hat{\mathrm{S}}_{11} = \frac{1}{R^2}(n_1 r_1 + n_2 r_2) \left[16 \frac{r_2^3}{R^2} - 4(5 - 2\nu) r_2 \right] - n_2 \left[4 \frac{r_2^2}{R^2} - 2(3 - 2\nu) \right] \,, \tag{4.13b}$$

$$\hat{\mathrm{S}}_{12} = n_1\left[4\frac{r_1^2}{R^2} - 2(1-2\nu)\right] - \frac{1}{R^2}n_2 r_1\left[16\frac{r_2^3}{R^2} - 4(1+2\nu)r_2\right]$$
$$-\frac{1}{R^2}n_1 r_2\left[16\frac{r_1^2 r_2}{R^2} - 4(1-2\nu)r_2\right], \tag{4.13c}$$
$$\hat{\mathrm{S}}_{21} = \hat{\mathrm{S}}_{12}, \tag{4.13d}$$
$$\hat{\mathrm{S}}_{22} = \frac{1}{R^2}n_1 r_2\left[16\frac{r_1^3}{R^2} - 4(3-2\nu)r_1\right] + \frac{1}{R^2}n_2 r_1\left[16\frac{r_1 r_2^2}{R^2} - 4(1-2\nu)r_2\right]$$
$$-n_2\left[4\frac{r_2^2}{R^2} + 2(1-2\nu)\right], \tag{4.13e}$$
$$\hat{\mathrm{S}}_{31} = \hat{\mathrm{S}}_{22}, \tag{4.13f}$$
$$\hat{\mathrm{S}}_{32} = \frac{1}{R^2}(n_1 r_1 + n_2 r_2)\left[16\frac{r_1^3}{R^2} - 4(5-2\nu)r_1\right] - n_1\left[4\frac{r_1^2}{R^2} - 2(3-2\nu)\right]. \tag{4.13g}$$

We remark that the normal is taken at $\overset{o}{M}$.

5. Somigliana formulae in dual system – outer region

By the outer region A_e we mean the region outside the contour $\mathcal{L}_0$. We shall assume that the stresses are constants at infinity. These are denoted by

$$t_{11}(\infty),\ t_{12}(\infty) = t_{21}(\infty) \ \text{ and } \ t_{22}(\infty).$$

We shall also assume that there is no rigid body rotation at infinity, that is,

$$\varphi_3(\infty) = 0\,. \tag{5.1}$$

Observe that the strains obtainable from the stresses at infinity via Hook's law are compatible for they satisfy the compatibility condition (2.3). The corresponding stress functions are of the form

$$\tilde{\mathfrak{u}}_\lambda(Q) = \epsilon_{\alpha 3\rho}\xi_\alpha t_{\lambda\rho}(\infty) + c_\lambda(\infty) \tag{5.2}$$

where $c_\lambda(\infty)$ is a constant vector to which there belong no stresses. Further let

$$\tilde{\mathfrak{u}}_3(Q) = -\varphi_3(\infty) = 0\,. \tag{5.3}$$

When deriving the dual Somigliana formula for the outer region A_e, we shall follow the line of thought of the previous section with an emphasis placed on the difference. It is assumed again that $\overset{*}{\mathfrak{u}}_k$ is the elastic state described by the fundamental solutions (3.11) and (3.12). Further, $\mathfrak{u}_k$ is also an elastic state arbitrary at finite but it is to meet the conditions

$$\mathfrak{u}_\kappa = \tilde{\mathfrak{u}}_\lambda \ \text{ and } \ \mathfrak{u}_3 = \tilde{\mathfrak{u}}_3 = 0$$

at infinity. Depending on the location of the point Q, we distinguish three cases in the same way as we did for the inner region A_i. It is assumed that the origin O is within the region A_i.

1. If $Q \in A_e$, we shall consider the triple connected region A'_e bounded by the contours $\mathcal{L}_0$, $\mathcal{L}_\varepsilon$ and the circle $\mathcal{L}_R$ with radius ${}_eR$ and center at O. Here $\mathcal{L}_\varepsilon$ is the contour of the neighborhood A_ε of Q with radius R_ε while ${}_eR$ is sufficiently large to involve both $\mathcal{L}_0$, and $\mathcal{L}_\varepsilon$. In addition, A_ε is to lie wholly in A'_e. Now we apply the dual Somigliana identity to the region A'_e and take the limit of the resulting equation

$$\oint_{\mathcal{L}_o} [\mathfrak{T}_{k\lambda}(\overset{o}{M},Q)\mathfrak{u}_\lambda(\overset{o}{M}) - \mathfrak{U}_{k\lambda}(\overset{o}{M},Q)\mathfrak{t}_\lambda(\overset{o}{M})]\, ds_{\overset{o}{M}} \tag{5.4}$$
$$+ \oint_{\mathcal{L}_\varepsilon} [\mathfrak{T}_{k\lambda}(\overset{o}{M},Q)\mathfrak{u}_\lambda(\overset{o}{M}) - \mathfrak{U}_{k\lambda}(\overset{o}{M},Q)\mathfrak{t}_\lambda(\overset{o}{M})]\, ds_{\overset{o}{M}}$$
$$+ \oint_{\mathcal{L}_R} [\mathfrak{T}_{k\lambda}(\overset{o}{M},Q)\mathfrak{u}_\lambda(\overset{o}{M}) - \mathfrak{U}_{k\lambda}(\overset{o}{M},Q)\mathfrak{t}_\lambda(\overset{o}{M})]\, ds_{\overset{o}{M}} = 0$$

as $R_\varepsilon \longrightarrow 0$ and ${}_eR \longrightarrow \infty$. As regards the sum of the first two integrals observe, that the limit is formally the same as that of the integrals in (4.4):

$$\oint_{\mathcal{L}_o} \cdots + \lim_{R_\varepsilon \longrightarrow 0} \oint_{\mathcal{L}_\varepsilon} \cdots = \mathfrak{u}_k(Q) + \oint_{\mathcal{L}_o} [\mathfrak{T}_{k\lambda}(\overset{o}{M},Q)\mathfrak{u}_\lambda(\overset{o}{M}) - \mathfrak{U}_{k\lambda}(\overset{o}{M},Q)\mathfrak{t}_\lambda(\overset{o}{M})]\, ds_{\overset{o}{M}}\,. \tag{5.5}$$

For the limit of the third integral we obtain

$$\lim_{{}_eR \longrightarrow \infty} \oint_{\mathcal{L}_R} \cdots = -\tilde{\mathfrak{u}}_k(Q)\,. \tag{5.6}$$

By making use of the results (5.5) and (5.6) we shall find from (5.4) for the first dual Somigliana identity on the outer region that

$$\mathfrak{u}_k(Q) = \tilde{\mathfrak{u}}_k(Q) + \oint_{\mathcal{L}_o} \mathfrak{U}_{k\lambda}(\overset{o}{M},Q)\mathfrak{t}_\lambda(\overset{o}{M})\, ds_{\overset{o}{M}} - \oint_{\mathcal{L}_o} \mathfrak{T}_{k\lambda}(\overset{o}{M},Q)\mathfrak{u}_\lambda(\overset{o}{M})\, ds_{\overset{o}{M}}\,. \tag{5.7}$$

Remark 11.: Derivation of the relation (5.7) requires long formal transformations. First one has to approximate $\mathfrak{U}_{k\lambda}$ and $\mathfrak{T}_{k\lambda}$ with one series in terms of ${}_eR$ to the power $1, 0, -1, -2$ etc. This transformation relies upon the use of the relations:

$$x_\alpha(\overset{o}{M}) = {}_eRn_\alpha(\overset{o}{M}), \qquad r_\alpha(\overset{o}{M},Q) = x_\alpha(\overset{o}{M}) - \xi_\alpha(Q) \tag{5.8a}$$

$$\frac{1}{R} = \frac{1}{{}_eR}\left(1 + \frac{n_\alpha(\overset{o}{M})\xi_\alpha(Q)}{{}_eR} - \frac{1}{2}\frac{\xi_\alpha(Q)\xi_\alpha(Q)}{{}_eR^2} + \cdots\right) \tag{5.8b}$$

$$\ln\frac{1}{R} = \ln\frac{1}{{}_eR} + \frac{n_\alpha(\overset{o}{M})\xi_\alpha(Q)}{{}_eR} - \frac{1}{2}\frac{\xi_\alpha(Q)\xi_\alpha(Q)}{{}_eR^2} + \cdots \tag{5.8c}$$

$$\frac{r_\alpha r_\beta}{{}_eR^2} \approx n_\alpha n_\beta + 2n_\alpha n_\beta\frac{n_\psi\xi_\psi}{{}_eR} - \frac{1}{{}_eR}\left(\xi_\alpha n_\beta + n_\alpha\xi_\beta\right) + \cdots \quad . \tag{5.8d}$$

Further one has also to utilize the following:

(a) For the outward unit normal and unit tangent in terms of φ, we may write

$$n_a = (\sin\varphi, \cos\varphi)\,, \qquad \tau_\alpha = (-\cos\varphi, \sin\varphi) \tag{5.9a}$$

(φ is the polar angle).

(b) For ${}_eR \longrightarrow \infty$ – see (3.17), (2.15), (2.14) and (2.2) –

$$\mathfrak{t}_\lambda(\overset{o}{M}) = -\frac{du_\lambda}{ds} = \tau_\kappa e_{\kappa\lambda} = \tau_\kappa \frac{1}{2\mu}\left(t_{\kappa\lambda}(\infty) - \nu t_{\psi\psi}(\infty)\delta_{\kappa\lambda}\right) \tag{5.9b}$$

and

$$ds_{\overset{o}{M}} = {}_eRd\varphi\,. \tag{5.9c}$$

(c) Since the stresses (consequently the strains as well) tend to constant value as ${}_eR \longrightarrow \infty$ the coefficients of ${}_eR$ always assume the form: an expression constant at infinity and multiplied by

$$\int_0^{2\pi} \sin^n\varphi \cos^k\varphi \, d\theta$$

where the powers n and k are natural numbers and depend on the term considered, but the integral is of zero value.

(d) The structure of the terms being the coefficients of ${}_eR$ to the power zero is similar, but they involve ξ_α and the trigonometric integrals are not necessarily equal to zero.

(e) It holds that

$$\oint_{\mathcal{L}_R} \mathfrak{T}_{\kappa\lambda}(\overset{o}{M}, Q)\, ds_{\overset{o}{M}} = -\delta_{\kappa\lambda}\,. \tag{5.9d}$$

(The same relation holds for any simply connected contour provided that Q is an inner point.)

For keeping the extent of the paper below a reasonable limit we have omitted the transformations leading to (5.7).

2. If $Q = \overset{o}{Q} \in \partial A = \mathcal{L}_o$, we shall consider the double connected region A'_e bounded by $\mathcal{L}'_o$, $\mathcal{L}'_\varepsilon$ and $\mathcal{L}_R$ where $\mathcal{L}'_o$ is the part of $\mathcal{L}_o$ that is left after the removal of A_ε and $\mathcal{L}'_\varepsilon$ is the part of $\mathcal{L}_\varepsilon$ that lies within A_e. Applying again the dual Somigliana identity to A'_e and taking the limit as as $R_\varepsilon \longrightarrow 0$ and ${}_eR \longrightarrow \infty$, we get the second dual Somigliana relation for the outer region A_e:

$$c_{\kappa\lambda}(\overset{o}{Q})\mathfrak{u}_\lambda(\overset{o}{Q}) = \tilde{\mathfrak{u}}_\kappa(Q) + \oint_{\mathcal{L}_o} \mathfrak{U}_{\kappa\lambda}(\overset{o}{M}, \overset{o}{Q})\mathfrak{t}_\lambda(\overset{o}{M})\, ds_{\overset{o}{M}} - \oint_{\mathcal{L}_o} \mathfrak{T}_{\kappa\lambda}(\overset{o}{M}, \overset{o}{Q})\mathfrak{u}_\lambda(\overset{o}{M})\, ds_{\overset{o}{M}}\,. \tag{5.10}$$

Remark 12.: The integral equation (5.10) with unknowns $\mathfrak{u}_\lambda(\overset{o}{M})$ on $\mathcal{L}_u$ and $\mathfrak{t}_\lambda(\overset{o}{M})$ on $\mathcal{L}_t$ is that of the direct method for outer regions.

Remark 13.: We have omitted again the details since the limit of the integral on $\mathcal{L}_R$ is the same as that for $Q \in A_e$ while the other terms can be derived letter by letter in the same way as for the integral equation (4.9).

3. It is obvious on the basis of all that has been said above that for $Q \in A_i$ the third dual Somigliana relation for the outer region is of the form

$$0 = \tilde{\mathfrak{u}}_k(Q) + \oint_{\mathcal{L}_o} \mathfrak{U}_{\kappa\lambda}(\overset{o}{M}, Q)\mathfrak{t}_\lambda(\overset{o}{M})\, ds_{\overset{o}{M}} - \oint_{\mathcal{L}_o} \mathfrak{T}_{\kappa\lambda}(\overset{o}{M}, Q)\mathfrak{u}_\lambda(\overset{o}{M})\, ds_{\overset{o}{M}} \,. \quad (5.11)$$

Let

$$\tilde{\mathfrak{s}}_k = (t_{11}(\infty), t_{12}(\infty), t_{22}(\infty)) \,. \quad (5.12)$$

By repeating the line of thought leading to (4.11) we obtain the formula for the stresses at the internal points of the outer region A_e:

$$\mathfrak{s}_k(Q) = \tilde{\mathfrak{s}}_k(Q) + \oint_{\mathcal{L}_o} \mathrm{D}_{k\lambda}(\overset{o}{M}, Q)\mathfrak{t}_\lambda(\overset{o}{M})\, ds_{\overset{o}{M}} - \oint_{\mathcal{L}_o} \mathrm{S}_{k\lambda}(\overset{o}{M}, Q)\mathfrak{u}_\lambda(\overset{o}{M})\, ds_{\overset{o}{M}} \quad (5.13)$$

where $\mathrm{D}_{k\lambda}(\overset{o}{M}, Q)$ and $\mathrm{S}_{k\lambda}(\overset{o}{M}, Q)$ are given by (4.12a),...,(4.13g).

6. Examples

We have applied the usual and well known procedure – see for instance [13] – for the solution of the boundary integral equation of the direct method (4.9). The program was written in Fortran 90.

In what follows we detail the main features of the algorithm.

We have used partially discontinuous quadratic elements by mapping the element onto the interval $\eta \in [-1, 1]$. The corresponding shape functions are Lagrange polynomials

$$\begin{aligned} N^1(\eta) &= \frac{1}{(\eta^1 - \eta^2)(\eta^1 - \eta^3)}(\eta - \eta^2)(\eta - \eta^3) \,, \\ N^2(\eta) &= \frac{1}{(\eta^2 - \eta^3)(\eta^2 - \eta^1)}(\eta - \eta^3)(\eta - \eta^1) \,, \\ N^3(\eta) &= \frac{1}{(\eta^3 - \eta^1)(\eta^3 - \eta^2)}(\eta - \eta^1)(\eta - \eta^2) \end{aligned} \quad (6.1)$$

with local nodal points η^1, η^2 and η^3 where $\eta^1 = -1$ and $-1 < \eta^2 < \eta^3 < 1$ are regarded as previously fixed parameters if the discontinuity occurs at $\eta = 1$ while $\eta^3 = 1$ and $-1 < \eta^1 < \eta^2 < 1$ are regarded again as previously fixed parameters if the discontinuity occurs at $\eta = -1$. For $\eta^1 = -1$ $\eta^2 = 0$ and $\eta^3 = 1$ the above polynomials give the usual isoparametric approximation.

Let n_{bn} be the number of nodal points. Further let n_{be} be the number of boundary elements. The elements are denoted by $\mathcal{L}_e$ – $e = 1, \dots, n_{be}$.

Let

$$\mathbf{u}_j = \begin{bmatrix} \mathfrak{u}_1^j \\ \mathfrak{u}_2^j \end{bmatrix} \qquad \text{and} \qquad \mathbf{t}_j = \begin{bmatrix} \mathfrak{t}_1^j \\ \mathfrak{t}_2^j \end{bmatrix} \qquad j = 1, \dots, n_{bn} \quad (6.2)$$

be the stress functions and the displacement derivative $-du_\lambda/ds$ at the nodal point j. The matrices of the stress functions $\mathbf{u}$ and that of the displacement derivatives $\mathbf{t}$

are defined by

$$\mathbf{u}^T = [\underbrace{\mathfrak{u}_1^1\,\mathfrak{u}_2^1}_{\mathbf{u}_1^T}|\underbrace{\mathfrak{u}_1^2\,\mathfrak{u}_2^2}_{\mathbf{u}_2^T}|\ \dots\ |\underbrace{\mathfrak{u}_1^{n_{bn}}\,\mathfrak{u}_2^{n_{bn}}}_{\mathbf{u}_{n_{bn}}^T}]\,, \tag{6.3a}$$

$$\mathbf{t}^T = [\underbrace{\mathfrak{t}_1^1\,\mathfrak{t}_2^1}_{\mathbf{t}_1^T}|\underbrace{\mathfrak{t}_1^2\,\mathfrak{t}_2^2}_{\mathbf{t}_2^T}|\ \dots\ |\underbrace{\mathfrak{t}_1^{n_{bn}}\,\mathfrak{t}_2^{n_{bn}}}_{\mathbf{t}_{n_{bn}}^T}]\,. \tag{6.3b}$$

Let there be constructed a function $a(j,e)$ giving the local node number of the node on element e with global node number j. For our latter considerations we introduce the integrals

$$\hat{\mathbf{h}}_{ij} = \left[\sum_{e\in j}\int_{\mathcal{L}_e}\mathfrak{T}_{\kappa\lambda}(Q_i,\eta)N^{a(j,e)}(\eta)J(\eta)\,d\eta\right] \tag{6.4}$$

and

$$\mathbf{b}_{ij} = \left[\sum_{e\in j}\int_{\mathcal{L}_e}\mathfrak{U}_{\kappa\lambda}(Q_i,\eta)N^{a(j,e)}(\eta)J(\eta)\,d\eta\right] \tag{6.5}$$

where the summation is taken over those elements containing the nodal point with number j, Q_i is the i-th nodal point (collocation point) and $J(\eta)$ is the Jacobian. With the notations (6.2),…,(6.5),

$$\mathbf{c}_{ii} = [c_{\kappa\lambda}(Q_i)] \tag{6.6}$$

and

$$\mathbf{h}_{ij} = \begin{cases}\hat{\mathbf{h}}_{ii}+\mathbf{c}_{ii} & \text{ha}\quad i=j\\ \hat{\mathbf{h}}_{ij} & \text{ha}\quad i\neq j\end{cases} \tag{6.7}$$

it follows from the second dual Somigliana formula (4.9) taken at the collocation point $\overset{o}{Q} = Q_i$ that

$$\begin{bmatrix}\mathbf{h}_{i1} & \mathbf{h}_{i2} & \cdots & \mathbf{h}_{in_{bn}}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\\ \cdots\\ \mathbf{u}_{n_{bn}}\end{bmatrix} = \begin{bmatrix}\mathbf{b}_{i1} & \mathbf{b}_{i2} & \cdots & \mathbf{b}_{in_{bn}}\end{bmatrix}\begin{bmatrix}\mathbf{t}_1\\ \mathbf{t}_2\\ \cdots\\ \mathbf{t}_{n_{bn}}\end{bmatrix}\,,$$

$$i = 1,\dots,n_{bn} \tag{6.8}$$

After uniting these equations we have

$$\begin{bmatrix}\mathbf{h}_{11} & \mathbf{h}_{12} & \cdots & \mathbf{h}_{1n_{bn}}\\ \mathbf{h}_{21} & \mathbf{h}_{22} & \cdots & \mathbf{h}_{2n_{bn}}\\ \dots & \dots & \dots & \dots\\ \mathbf{h}_{n_{bn}1} & \mathbf{h}_{n_{bn}2} & \cdots & \mathbf{h}_{n_{bn}n_{bn}}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\\ \cdots\\ \mathbf{u}_{n_{bn}}\end{bmatrix} = \begin{bmatrix}\mathbf{b}_{11} & \mathbf{b}_{12} & \cdots & \mathbf{b}_{1n_{bn}}\\ \mathbf{b}_{21} & \mathbf{b}_{22} & \cdots & \mathbf{b}_{2n_{bn}}\\ \dots & \dots & \dots & \dots\\ \mathbf{b}_{n_{bn}1} & \mathbf{b}_{n_{bn}2} & \cdots & \mathbf{b}_{n_{bn}n_{bn}}\end{bmatrix}\begin{bmatrix}\mathbf{t}_1\\ \mathbf{t}_2\\ \cdots\\ \mathbf{t}_{n_{bn}}\end{bmatrix} \tag{6.9}$$

or in a more compact form

$$\mathbf{Hu} = \mathbf{Bt}\,. \tag{6.10}$$

After solving the above system of linear equations we have the nodal values of the unknown stress functions $\mathfrak{u}_\lambda$ on $\mathcal{L}_u$ and the nodal values of the unknown displacement derivatives $\mathfrak{t}_\lambda$ on $\mathcal{L}_t$.

In the knowledge of the nodal values stresses at the internal points are computed by using (4.11). Stresses on $\mathcal{L}_u$ are computed element by element by substituting the local approximation of the stress functions into (2.5).

Since there belong no stresses to the constant stress functions they are determined with the accuracy of a constant vector. Consequently

$$t_\lambda = -\frac{du_\lambda}{ds} = -\varphi_3 \frac{dx^\kappa}{ds} \epsilon_{3\kappa\lambda}$$

where φ_3 is the rigid body rotation which is to be constant if there are no stresses and strains. Therefore it can be set to zero. If this is the case then $\mathbf{t} = \mathbf{0}$ and if we take the constant stress functions as $u_k = 1$ $(k = 1, \ldots, n_{bn})$, then we have

$$\sum_{j=1}^{2n_{bn}} H_{ij} = 0 \quad \text{or, which is the same thing,} \quad H_{ii} = -\sum_{\substack{j=1 \\ (i \neq j)}}^{2n_{bn}} H_{ij} \qquad i = 1, 2, \ldots, 2n_{bn} \,, \tag{6.11}$$

where H_{ij} is an element of the matrix $\mathbf{H}$. By using this property one can avoid the numerical integration of strongly singular integrals.

If the region under consideration is an outer one, then there are some changes in the final equation system. Let the matrix $\tilde{\mathbf{u}}$ be defined by

$$\tilde{\mathbf{u}}^T = [\underbrace{\tilde{\mathfrak{u}}_1^1\, \tilde{\mathfrak{u}}_2^1}_{\tilde{\mathbf{u}}_1^T} | \underbrace{\tilde{\mathfrak{u}}_1^2\, \tilde{\mathfrak{u}}_2^2}_{\tilde{\mathbf{u}}_2^T} | \ldots | \underbrace{\tilde{\mathfrak{u}}_1^{n_{bn}}\, \tilde{\mathfrak{u}}_2^{n_{bn}}}_{\tilde{\mathbf{u}}_{n_{bn}}^T}] \tag{6.12}$$

where $\tilde{\mathbf{u}}_j$ is the matrix that involves $\tilde{\mathfrak{u}}_\kappa$ at the nodal point Q_j $(j = 1, \ldots, n_{bn})$. With this notation the equation system to be solved for the unknown nodal values takes the form

$$\mathbf{H}\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{B}\mathbf{t}. \tag{6.13}$$

Computation of strongly singular integrals can be avoided if we use the relation

$$H_{ii} = -\sum_{\substack{j=1 \\ (i \neq j)}}^{2n_{bn}} H_{ij} + 1 \qquad i = 1, 2, \ldots, 2n_{bn} \,. \tag{6.14}$$

Equation (6.14) can be established in the same way as (6.11). c_ρ in $\tilde{\mathbf{u}}_j$ – see (5.2) – is set to zero.

Three examples are presented. The region under consideration including its matter, is the same for the first two cases. $r_0 = 10$[mm], $\mu = 8 \cdot 10^4$[MPa], $\nu = 0.3$

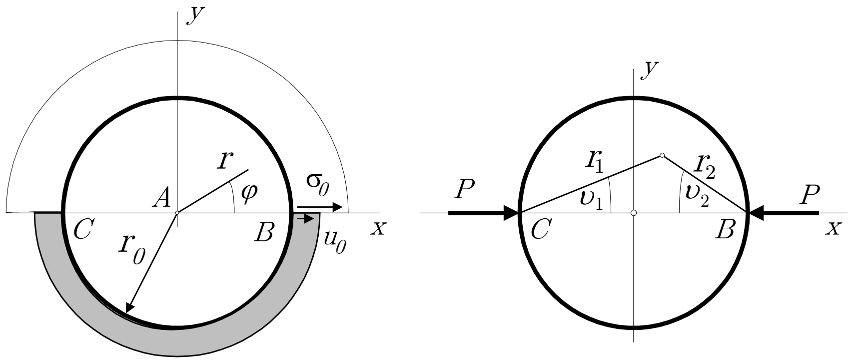

Figure 3.

Problem 1. On arc BC the radial stress (normal stress) is $\sigma_o = 100$ [MPa] (there are no shear stresses). On arc CB the radial displacement is $u_o = (1-2\nu)\sigma_o r_o/2\mu$ (there is no tangential displacement). One can check with ease that these values determine a homogeneous state of stress of the region. The exact solutions are given by the equations

$$\mathcal{F}_1 = \mathcal{F}_x = \sigma_o y = \sigma_o r \sin\varphi\ , \qquad \mathcal{F}_2 = \mathcal{F}_y = -\sigma_o x = -\sigma_o r\cos\varphi\ ,$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_o\ , \qquad \tau_{xy} = 0\ ,$$

$$u_x = \frac{1-2\nu}{2\mu}\sigma_o x = \frac{1-2\nu}{2\mu}\sigma_o r\sin\varphi\ ,$$

$$u_y = \frac{1-2\nu}{2\mu}\sigma_o y = \frac{1-2\nu}{2\mu}\sigma_o r\cos\varphi$$

where r and φ are polar coordinates. On arc BC and CB

$$\mathfrak{u}_x = \mathcal{F}_x = \sigma_o r_o \sin\varphi\ , \qquad \mathfrak{u}_y = \mathcal{F}_y = -\sigma_o r_o\cos\varphi$$

and

$$-\mathfrak{t}_x = \frac{du_x}{ds} = \frac{1-2\nu}{2\mu}\sigma_o\sin\varphi\ , \qquad -\mathfrak{t}_y = \frac{du_y}{ds} = \frac{1-2\nu}{2\mu}\sigma_o\cos\varphi$$

are the boundary conditions. The contour was divided into 16 equidistant elements. The table below contains the numerical results for the stresses

x [mm]	y [mm]	σ_{xx} [MPa]	τ_{xy} [MPa]	σ_{yy} [MPa]
-7.50	0.00	99.99927	0.0001113	99.99983
-5.00	0.00	99.99912	0.0000478	99.99988
-2.50	0.00	99.99917	0.0000182	99.99983
0.00	0.00	99.99918	0.0000000	99.99982
2.50	0.00	99.99917	0.0000182	99.99983
5.00	0.00	99.99912	0.0000478	99.99988
7.50	0.00	99.99927	0.0001112	99.99983
7.50	5.00	99.98317	0.0048330	100.00853
5.00	7.50	100.00854	0.0048269	99.98308
9.00	1.00	99.96938	0.0122755	100.02786

Problem 2. The region is subjected to a pair of compressive forces with magnitude 100.0 N/mm. Consequently the boundary conditions on the arcs AB and BC are

$$\mathcal{F}_x = P = -100\,, \qquad \mathcal{F}_y = 0$$

and

$$\mathcal{F}_x = 0\,, \qquad\qquad \mathcal{F}_y = 0\,,$$

respectively. With the notations of Figure 4

$$\begin{aligned}\sigma_{xx} &= \frac{2P}{\pi}\left[\frac{\cos^3\vartheta_1}{r_1} + \frac{\cos^3\vartheta_2}{r_2}\right] - \frac{P}{\pi r_o}\,,\\ \tau_{xy} &= -\frac{2P}{\pi}\left[\frac{\sin\vartheta_1\cos^2\vartheta_1}{r_1} - \frac{\sin\vartheta_2\cos^2\vartheta_2}{r_2}\right]\,,\\ \sigma_{yy} &= \frac{2P}{\pi}\left[\frac{\sin^2\vartheta_1\cos\vartheta_1}{r_1} + \frac{\sin^2\vartheta_2\cos\vartheta_2}{r_2}\right] - \frac{P}{\pi r_o}\end{aligned}$$

are the exact solutions [14]. Figures 4 to 6 represent the exact and the numerical solutions. The latter is denoted by diamonds. In this case the contour was divided into 40 equidistant elements. The pairs of elements that meet at A and B are partially discontinuous.

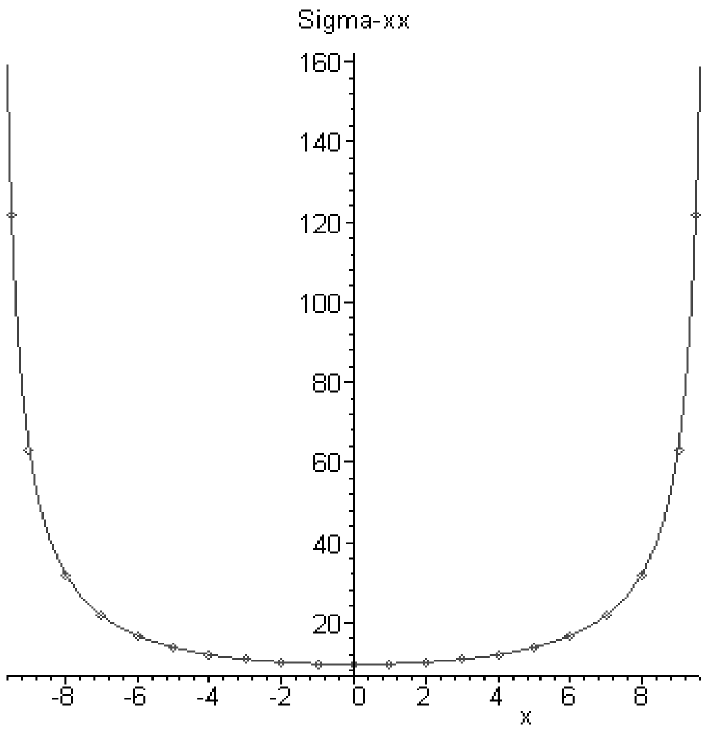

Figure 4. Exact and numerical solution – σ_{xx} along the horizontal diameter

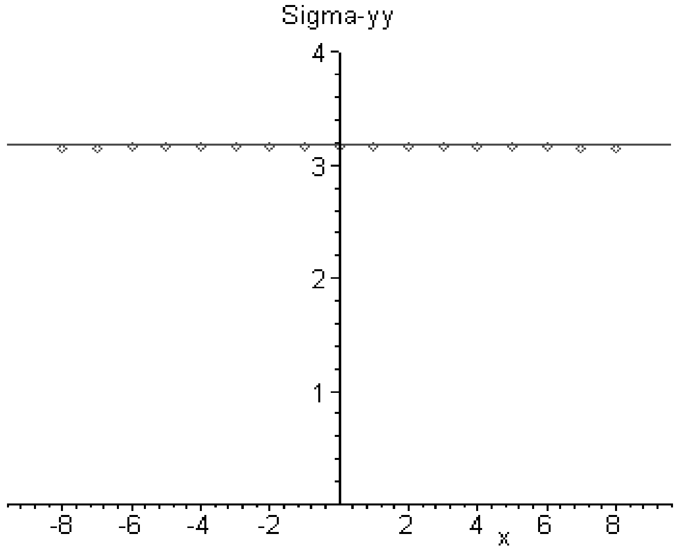

Figure 5. Exact and numerical solution – σ_{yy} along the horizontal diameter

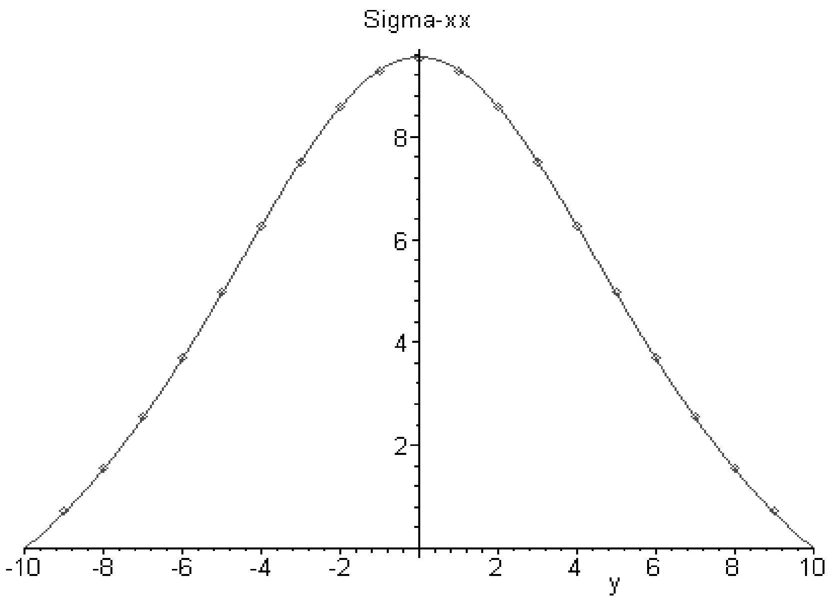

Figure 6. Exact and numerical solution – σ_{xx} along the vertical diameter

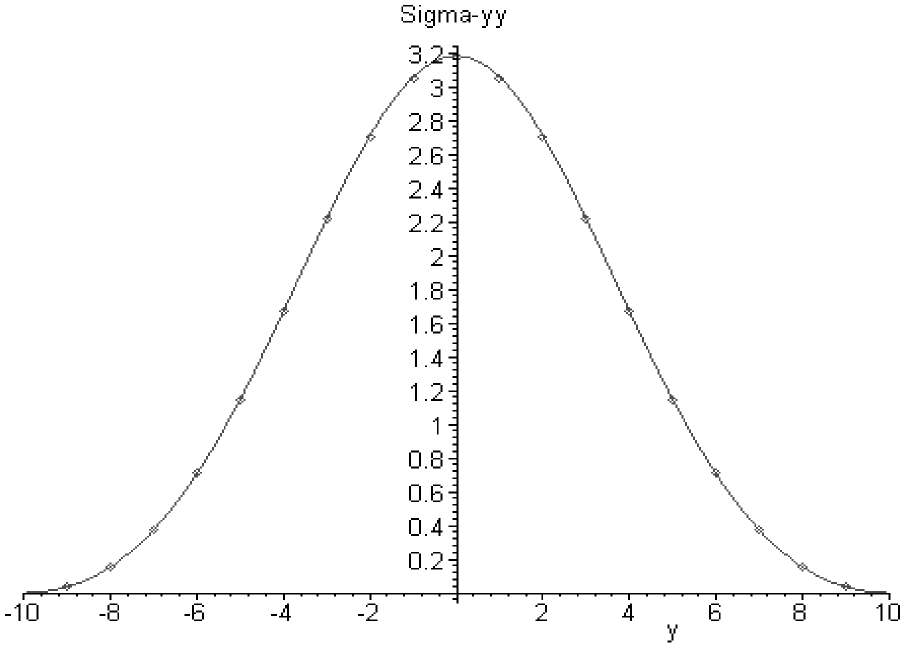

Figure 7. Exact and numerical solution – σ_{yy} along the vertical diameter

Problem 3. Though the contour $\mathcal{L}_o$ and the material are the same as in the previous examples the region under consideration is the outer one for which a constant stress state $\sigma_{xx}(\infty) = 100$[MPa], $\sigma_{xy}(\infty) = \sigma_{yx}(\infty) = \sigma_{yy}(\infty) = 0$ is prescribed at infinity.

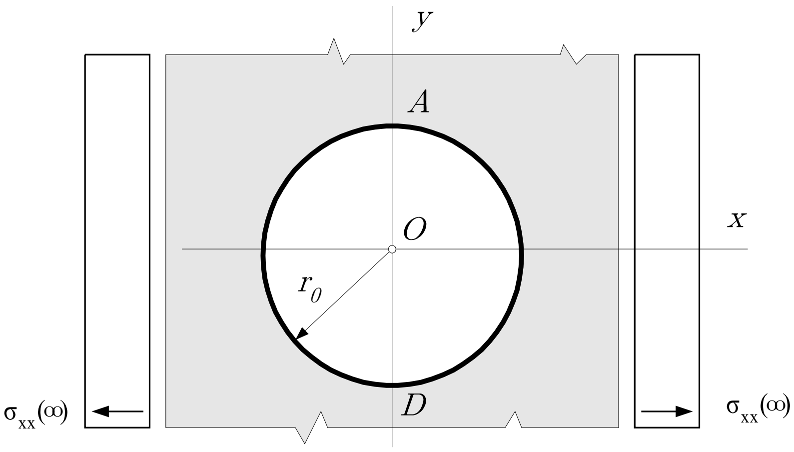

Figure 8. Outer region bounded by a circle with radius $r_o = 10$ [mm] and centered at O

It is well known that the formulae

$$\sigma_{rr} = \frac{\sigma_{xx}(\infty)}{2}\left[\left(1-\frac{r_o^2}{r^2}\right)+\left(1+\frac{3r_o^4}{r^4}-\frac{4r_o^2}{r^2}\right)\cos 2\varphi\right] ,$$

$$\sigma_{\varphi\varphi} = \frac{\sigma_{xx}(\infty)}{2}\left[\left(1+\frac{r_o^2}{r^2}\right)-\left(1+\frac{3r_o^4}{r^4}\right)\cos 2\varphi\right] ,$$

$$\sigma_{r\varphi} = \frac{\sigma_{xx}(\infty)}{2}\left[\left(1-\frac{3r_o^4}{r^4}+\frac{2r_o^2}{r^2}\right)\sin 2\varphi\right]$$

written in polar coordinates give the exact solution to this problem [15], [14]. The table below shows both the stresses we computed and the exact solution on the y axis. The contour was divided into 16 equidistant element.

x [mm]	y [mm]	σ_{xx} [MPa]	τ_{xy} [MPa]	σ_{yy} [MPa]
0.00	10.00	300.0395	0.0000000	0.001035
		300.0000	0.0000000	0.000000
0.00	11.00	243.7623	0.0000000	21.51840
		243.7743	0.0000000	21.51494
0.00	12.00	207.0554	0.0000000	31.82829
		207.0602	0.0000000	31.82870
0.00	13.00	182.1018	0.0000000	36.23794
		182.1049	0.0000000	36.23823
0.00	14.00	164.5539	0.0000000	37.48413
		164.5564	0.0000000	37.48438
0.00	15.00	151.8498	0.0000000	37.03671
		151.8518	0.0000000	37.03704

7. Concluding remarks

In accordance with our aims we have clarified what the supplementary conditions of single valuedness are for a class of mixed boundary value problems in the dual system of plane elasticity assuming multiply connected domains.

The fundamental solutions for the stress functions of order one have also been constructed. In the knowledge of the fundamental solutions we have established the dual Somigliana relations both for inner regions and for outer ones, which involve the equations of the direct method. It has been shown that the system matrix **H** has the same properties as in the primal system that is the sum of the elements in a row is equal to zero (inner region) or to one (outer region). A program has been developed in Fortran 90 for the numerical solution by using partially discontinuous quadratic boundary elements. The three examples illustrate the applicability of the algorithm.

Two advantages of the algorithm are worthy of mention (a) calculation of stresses requires the knowledge of the first derivatives of stress functions (b) concentrated forces can be handled. It is, however a disadvantage that the supplementary conditions of single valuedness should be taken into account on multiply connected domains. The present program has not been capable of handling multiply connected domains.

It can be shown, though the proof is not presented here, that the integrand in the boundary integral equations is divergence free. Therefore it is possible to develop the boundary contour method in a dual system as well [16].

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